These notes are from https://mathematicaster.org/teaching/graphs2022/extra_02-24.pdf

Here is a slightly different way to understand cut-vertices and blocks than that used in the book.

Definition 1. *Let G be a graph. We define the relation $\mathcal{C}$ on $E(G)$ by $e\,\mathcal{C}\,s$ iff $e = s$ or there is a cycle in G which uses both e and s.*

HW6.6 asks you to verify that $\mathcal{C}$ is an equivalence relation on $E(G)$.

Lemma 2. *Let G be a graph and fix $v \in V(G)$. The vertex v is a cut-vertex of G if and only if there are edges $e, s \in E(G)$, both containing v, with $(e, s) \notin \mathcal{C}$.*

Proof. Without loss of generality, we may suppose that G is connected (why?).

Let $E_v = \{e \in E(G) : e \ni v\}$ be the set of edges of G which contain the vertex v. If $|E_v| \leq 1$, then $\deg v \leq 1$ and so v cannot be a cut-vertex and every element of E_v is trivially related in $\mathcal{C}$. Thus, we may suppose that $|E_v| \geq 2$.

Fix any $e \neq s \in E_v$; suppose that $e = uv$ and $s = wv$, so $u \neq w$. We claim that there is a u-w path in $G - v$ if and only if $e\,\mathcal{C}\,s$. For one direction, if $(u = u_0, \dots, u_k = w)$ is a u-w path in $G - v$, then $(u = u_0, \dots, u_k = w, v)$ is a cycle containing both uv and wv since v does not belong to $\{u_0, \dots, u_k\}$; thus $e\,\mathcal{C}\,s$. On the other hand, if $e\,\mathcal{C}\,s$, then there is a cycle C in G containing both e and s since $e \neq s$. Since e and s are both incident to v, we can label this cycle as $(w, v, u, c_1, \dots, c_k)$ for some $k \geq 0$. But then $(u, c_1, \dots, c_k, w)$ is a u-w path in $G - v$.

Now, suppose that $G - v$ has connected components $G_1, \dots, G_k$. For each $i \in [k]$, set $N_i = \{u \in V(G_i) : uv \in E(G)\}$. Note that $N_i \neq \varnothing$ for each i since G is connected and that $N(v) = \bigsqcup_{i=1}^{k} N_i$. From above, we know that $u_i v\,\mathcal{C}\,u_i' v$ for all $u_i, u_i' \in N_i$ and that $(u_i v, u_j v) \notin \mathcal{C}$ for all $i \neq j$ and all $u_i \in N_i$, $u_j \in N_j$. Thus, $k = 1$ (i.e. v is not a cut-vertex) if and only if $e\,\mathcal{C}\,s$ for all $e, s \in E_v$ which concludes the proof. □

Phrasing the above lemma differently: v is *not* a cut-vertex of G if and only if $e\,\mathcal{C}\,s$ for every $e, s \in E(G)$ which both contain v. We can now determine exactly when cut-vertices exist.

Theorem 3. *Let G be a connected graph on at least two vertices. G has no cut-vertices if and only if $\mathcal{C}$ has exactly one equivalence class.*

Proof. For ease of notation, set $E = E(G)$. Suppose that the equivalence classes of $\mathcal{C}$ are $\mathcal{E}_1, \dots, \mathcal{E}_k$.

($\Leftarrow$) Suppose that $\mathcal{C}$ has exactly one equivalence class, so $k = 1$. Then, because every pair of edges in E are related under $\mathcal{C}$, Lemma 2 implies that G has no cut-vertices.

($\Rightarrow$) We prove the contrapositive, so suppose that $k \neq 1$. We can't have $k = 0$ since then $E = \varnothing$ which is impossible for a connected graph on at least two vertices, so $k \geq 2$. Define a function $f\colon E \to [k]$ where $f(e) = i$ if and only if $e \in \mathcal{E}_i$. Since the $\mathcal{E}_i$'s are equivalence classes, we know that each $\mathcal{E}_i$ is non-empty and they partition E; hence f is well-defined and $f^{-1}(i) = \mathcal{E}_i$ for each $i \in [k]$. Note that $f(e) = f(s)$ if and only if $e\,\mathcal{C}\,s$.

To go further, recall the line graph $L(G)$, introduced in DS1.7, which has vertex-set E and es is an edge of $L(G)$ iff $|e \cap s| = 1$. Since G is connected, we know that $L(G)$ is connected (DS1.7.4). Now, f is a function from the vertex-set of $L(G)$ (which is E) to $[k]$. Additionally, f is *not* a constant function since $k \geq 2$ and $f^{-1}(i) = \mathcal{E}_i \neq \varnothing$ for each $i \in [k]$. Thus, HW2.2 implies that an edge es of $L(G)$ with $f(e) \neq f(s)$. Since es is an edge in $L(G)$, we have $|e \cap s| = 1$. Suppose that $e \cap s = \{v\}$. Now, $e, s \in E(G)$ both contain v and yet $(e, s) \notin \mathcal{C}$ since $f(e) \neq f(s)$. Lemma 2 then tells us that v is a cut-vertex of G. □

Phrasing the above theorem differently: A connected graph has no cut-vertices if and only if every pair of distinct edges live in some common cycle. It turns out that the same is true of vertices, provided we have enough vertices.

Theorem 4. *Let G be a connected graph on at least three vertices. G has no cut-vertices if and only if for every $u \neq v \in V(G)$, there is a cycle of G containing both u and v.*

Proof. ($\Leftarrow$) Fix any $v \in V(G)$; we need to show that $G - v$ is connected. Consider any $u \neq w \in V(G) \setminus \{v\}$, which can be done since G has at least three vertices. By assumption, there is a cycle C in G containing both u and w. Suppose that this cycle is $(u = u_1, \dots, u_k)$ where $u_\ell = w$ for some $\ell \in \{2, \dots, k\}$. Then both $(u_1, \dots, u_\ell)$ and $(u_1, u_k, \dots, u_{\ell+1}, u_\ell)$ are u-w paths in G which are interally disjoint. Thus, at least one of these does not use the vertex v and so there is a u-w path in $G - v$. In other words, $G - v$ is connected and so v is not a cut-vertex.

($\Rightarrow$) Consider any $u \neq v \in V(G)$. Since G is connected and has at least three vertices, we can therefore find some edges $e \neq s \in E(G)$ with $e \ni u$ and $s \ni v$. Since $e \neq s$ and G has no cut-vertices, Theorem 3 implies that e and s are contained together within some cycle of G. This same cycle contains the vertices u and v. □

Let G be a graph and suppose that $\mathcal{C}$ has equivalence classes $\mathcal{E}_1, \dots, \mathcal{E}_k$. For each $i \in [k]$, we can define a subgraph of G which has edge-set $\mathcal{E}_i$ and vertex-set $\bigcup_{e \in \mathcal{E}_i} e$ (that is, all vertices incident to some edge of $\mathcal{E}_i$). Such a subgraph is called a *block* of G. Denote the block formed from $\mathcal{E}_i$ by B_i. We make the following observations:

1. $B_1, \dots, B_k$ are edge-disjoint and every edge of G belongs to exactly one of $B_1, \dots, B_k$.
2. Each B_i is connected and has no cut-vertices.
3. $|E(B_i)| = 1$ or $|E(B_i)| \geq 3$.
4. Although $B_1, \dots, B_k$ are edge-disjoint, they can share vertices.
5. If v is not an isolated vertex, then v belongs to at least one block.[1]

Next time, we will understand exactly how blocks can share vertices.

The blocks $B_1, \dots, B_k$ are analogous to the connected components of a disconnected graph. The connected components are a decomposition of a general graph into connected chunks. The blocks are a decomposition of a general graph into extra-connected chunks.

[1]Some people would say that an isolated vertex is its own block. I, personally, don't like this, but I will ensure that this hiccup will never matter when it comes to any problem I ask you to solve. Just be aware of this discrepancy. If every you think I've erred and this issue does matter for some problem, please just ask.