These notes are from https://mathematicaster.org/teaching/graphs2022/extra_02-10.pdf

Here are two different (but very similar) proofs that a connected graph has a spanning tree. The ideas here can be very useful.

**Theorem 1.** *If $G$ is connected, then $G$ contains a spanning tree.*

*Proof #1.* Let $\mathcal{G}$ denote the set of all spanning subgraphs of $G$ which are connected. Observe that $\mathcal{G}$ is non-empty since certainly $G \in \mathcal{G}$. Therefore, let $T \in \mathcal{G}$ be any element with the fewest number of edges. We claim that $T$ is a spanning tree of $G$.

Firstly, since $T \in \mathcal{G}$, we know that $T$ is a spanning subgraph of $G$ and is connected; thus we need only show that $T$ is acyclic. Suppose for the sake of contradiction that $T$ contains a cycle; call it $C$. Fix any edge $e \in E(C)$ and consider $T' = T - e$. Since we did not modify the vertex set by removing the edge $e$, we know that $T'$ is still a spanning subgraph of $G$. Additionally, $T'$ is still connected since $e$ was chosen to be in a cycle of $T$. But this means that $T' \in \mathcal{G}$; a contradiction since $T'$ has strictly fewer edges than does $T$. □

*Proof #2.* Let $\mathcal{G}$ denote the set of all spanning subgraphs of $G$ which are acyclic. Observe that $\mathcal{G}$ is non-empty since $(V(G), \varnothing) \in \mathcal{G}$. Therefore, let $T \in \mathcal{G}$ be any element with the maximum number of edges. We claim that $T$ is a spanning tree of $G$.

Firstly, since $T \in \mathcal{G}$, we know that $T$ is a spanning subgraph of $G$ and is acyclic; thus we need only show that $T$ is connected. Suppose for the sake of contradiction that $T$ is disconnected; thus we can partition $V(T) = A \sqcup B$ with both $A$ and $B$ non-empty such that there are no edges of $T$ between $A$ and $B$. Since $V(T) = V(G)$ and $G$ is connected, there must be some edge $e \in E(G) \setminus E(T)$ such that $e$ has one end-point in $A$ and the other in $B$; consider $T' = T + e$. Firstly, $T'$ is still a spanning subgraph of $G$ since we did not modify the vertex set. Next, $T'$ is still acyclic since the edge $e$ must have had its end points in different connected components of $T$. But this means that $T' \in \mathcal{G}$; a contradiction since $T'$ has strictly more edges than does $T$. □

**Corollary 2.** *If $G$ is a connected graph on $n$ vertices, then $|E(G)| \geq n - 1$ with equality if and only if $G$ is a tree.*

*Proof.* Theorem 1 guarantees that $G$ has a spanning tree, call it $T$. Since $T$ is a tree on $n$ vertices, we know that $|E(T)| = n - 1$. Therefore, $n - 1 \leq |E(T)| \leq |E(G)|$ with equality if and only if $G = T$. □

Let's see another application of these ideas. The following theorem says that, in a connected graph, any set of edges can be extended to a spanning subgraph without introducing any extra cycles. The most common application of the theorem is that any acyclic set of edges can be extended to a spanning tree.

**Theorem 3.** *Let $G$ be a connected graph and let $S$ be any subset of edges of $G$. Then $G$ has a connected, spanning subgraph $H$ such that $E(H) \supseteq S$ and if $C$ is a cycle in $H$, then $E(C) \subseteq S$.*

Before we prove the theorem, notice that Theorem 1 follows as a corollary by taking $S = \varnothing$.

*Proof.* Let $\mathcal{G}$ denote the set of all $H$ such that

- $H$ is a spanning subgraph of $G$, and

- $H$ is connected, and
- $E(H) \supseteq S$.

We note that $\mathcal{G}$ is non-empty since $G \in \mathcal{G}$. Thus, let $H \in \mathcal{G}$ be any element with the fewest number of edges. We claim that $H$ is our desired subgraph.

Firstly, since $H \in \mathcal{G}$, we know that $H$ is a spanning subgraph of $G$, is connected and $E(H) \supseteq S$. So we need only show that $H$ does not have any extraneous cycles. Suppose that $C$ is a cycle in $H$ with $E(C) \not\subseteq S$. Therefore, there is some edge $e \in E(C) \setminus S$; consider $H' = H - e$. Since we did not modify the vertex set by removing $e$, $H'$ is a spanning subgraph of $G$. Additionally, $E(H') \supseteq S$ since $e \notin S$. Finally, $H'$ is still connected since $e$ was chosen to be in a cycle of $H$. But this means that $H' \in \mathcal{G}$; a contradiction since $H'$ has strictly fewer edges than does $H$. □