These notes are from https://mathematicaster.org/teaching/graphs2022/extra_02-01.pdf

In addition to Havel–Hakimi, there are other necessary and sufficient conditions for a sequence to be graphical. One of the most famous is the Erdős–Gallai Theorem. In this class, we prove only that the conditions are necessary. For a fairly short and sweet proof that the conditions are sufficient, look at the paper A short constructive proof of the Erdoős–Gallai characterization of graphic lists by Tripathi, Venugopalan and West.

Theorem 1 (Erdős–Gallai)**.** *A sequence $d_1 \geq \cdots \geq d_n$ is graphical if and only if $\sum_{i=1}^n d_i$ is even and for every $k \in [n]$,*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Proof. As mentioned, we will prove only that the conditions are necessary. To this end, suppose that $d_1 \geq \cdots \geq d_n$ is graphical and let $G = (V, E)$ be a realization. Label $V = \{v_1, \ldots, v_n\}$ where $\deg v_i = d_i$.

The fact that $\sum_{i=1}^n d_i$ is even follows directly from the handshaking lemma, so we focus on the latter condition. Fix any $k \in [n]$ and consider the set

$$\Omega = \big\{(x, y) \in \{v_1, \ldots, v_k\} \times V : xy \in E\big\}.$$

We begin by computing

$$|\Omega| = \sum_{i=1}^{k} |\{y \in V : v_i y \in E\}| = \sum_{i=1}^{k} \deg v_i = \sum_{i=1}^{k} d_i. \tag{1}$$

Next, we partition $\Omega = \Omega_1 \sqcup \Omega_2$ where

$$\Omega_1 = \big\{(x, y) \in \{v_1, \ldots, v_k\}^2 : xy \in E\big\}, \quad \text{and} \quad \Omega_2 = \big\{(x, y) \in \{v_1, \ldots, v_k\} \times \{v_{k+1}, \ldots, v_n\} : xy \in E\big\}.$$

Since this is a partition, we know that $|\Omega| = |\Omega_1| + |\Omega_2|$. We first bound

$$|\Omega_1| \leq |\{(x, y) \in \{v_1, \ldots, v_k\}^2 : x \neq y\}| = k(k-1). \tag{2}$$

Next we bound

$$|\Omega_2| = \sum_{i=k+1}^{n} |\{x \in \{v_1, \ldots, v_k\} : xv_i \in E\}| \leq \sum_{i=k+1}^{n} \min\{k, \deg v_i\} = \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{3}$$

Combining (1), (2) and (3) yields the claim. □