These notes are from https://mathematicaster.org/teaching/graphs2022/extra_01-27.pdf

Here's a cute fact that doesn't appear to be in the book:

Theorem 1. *If G is a connected graph wherein $\deg v$ is even for all $v \in V(G)$, then $G - e$ is connected for any $e \in E(G)$.*

Proof. Fix any edge $e = v_1v_2 \in E(G)$ and suppose for the sake of contradiction that $G - e$ is disconnected. If this is the case, then we can write $G - e = G_1 \sqcup G_2$ where G_i is connected and $v_i \in V(G_i)$ (why?) Now, observe that for any $v \in V(G) = V(G - e)$, we have

$$\deg_{G-e} v = \begin{cases} \deg_G v - 1 & \text{if } v \in e, \\ \deg_G v & \text{otherwise.} \end{cases}$$

Of course, since there are no edges between G_1 and G_2, we know that $\deg_{G_i} v = \deg_{G-e} v$ for all $v \in V(G_i)$. But we know that $\deg_{G_1} v_1$ is odd yet $\deg_{G_1} v$ is even for all $v \in V(G_1) \setminus \{v_1\}$! That is to say, G_1 has an odd number of odd-degree vertices; contradiction. □

Here's a careful proof of the theorem I messed up in class.

Theorem 2. *Fix an integer $r \geq 0$. If G is any graph with $\Delta(G) \leq r$, then G is an induced subgraph of an r-regular graph.*

We will require the following simple observation: If G is an induced subgraph of H and H is an induced subgraph of J, then G is an induced subgraph of J (i.e. this relation is transitive).

To prove the theorem, we actually prove the following equivalent statement by induction on k:

Theorem 3. *Fix an integer $r \geq 0$. For any integer $0 \leq k \leq r$, if G is any graph with $\Delta(G) \leq r$ and $\delta(G) = r - k$, then G is an induced subgraph of an r-regular graph.*

Proof. We begin with the base-case of $k = 0$. Here we have $r \geq \Delta(G) \geq \delta(G) = r - k = r$; in other words, G is an r-regular graph. Since G is an induced subgraph of itself, the claim follows.

Suppose now that $1 \leq k \leq r$ and suppose that $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ for ease of notation. Define $V' = \{v_1', \ldots, v_n'\}$ and build a new graph H as follows:

$$V(H) = V \sqcup V'$$
$$E(H) = E \sqcup \{v_i'v_j' : v_iv_j \in E\} \sqcup \{v_iv_i' : \deg_G(v_i) < r\}.$$

Observe that $H[V] = G$ and also that $H[V']$ is a "copy" of G (we will define this formally in a couple class periods). In particular, the former means that G is an induced subgraph of H.

We now consider the degrees of the graph H. Note that the edge v_iv_i' exists if and only if $\deg_G(v_i) < r$ and that $\deg_H(v_i) = \deg_H(v_i')$. Therefore,

$$\deg_H(v_i) = \deg_H(v_i') = \begin{cases} \deg_G(v_i) + 1 & \text{if } \deg_G(v_i) < r, \\ \deg_G(v_i) & \text{otherwise.} \end{cases}$$

In particular, since $\delta(G) = r - k < r$, we have $\delta(H) = \delta(G) + 1 = (r - k) + 1 = r - (k - 1)$. Additionally, since $\Delta(G) \leq r$ we know that $\Delta(H) \leq r$ as well; That is to say, H satisfies the hypotheses of the theorem with $(k-1)$ in place of k. We may therefore apply the induction hypothesis to find that there is some r-regular graph J which contains H as an induced subgraph. Since G is an induced subgraph of H, G is then also an induced subgraph of J, which establishes the claim. $\square$