These notes are from https://mathematicaster.org/teaching/graphs2022/extra_01-25.pdf

I was unhappy with the proof of the handshaking lemma given in the book (Theorem 2.1). Here's a much more careful proof. We'll use the following observation, which I encourage you to keep in mind:

If $X$ and $Y$ are finite sets and $\Omega \subseteq X \times Y$, then

$$|\Omega| = \sum_{x\in X} |\{y \in Y : (x, y) \in \Omega\}| = \sum_{y\in Y} |\{x \in X : (x, y) \in \Omega\}|.$$

**Theorem 1** (Handshaking Lemma)**.** *In a graph $G = (V, E)$*

$$\sum_{v\in V} \deg v = 2|E|$$

*Proof.* Consider the set

$$\widehat{E} = \{(u, v) \in V^2 : uv \in E\}.$$

Observe that if $uv \in E$, then both $(u, v)$ and $(v, u)$ are members of $\widehat{E}$. Intuitively, $|\widehat{E}| = 2|E|$. To actually prove this, consider the function $f \colon \widehat{E} \to E$ defined by $f(u, v) = uv$; note that $f$ is well-defined. Now, for any $uv \in E$, observe that the pre-image of $uv$ is $f^{-1}(uv) = \{(u, v), (v, u)\}$. Since these pre-images partition the domain of $f$, we thus have

$$|\widehat{E}| = \sum_{uv\in E} |f^{-1}(uv)| = \sum_{uv\in E} 2 = 2|E|.$$

Finally, we compute

$$2|E| = |\widehat{E}| = \sum_{v\in V} |\{u \in V : uv \in E\}| = \sum_{v\in V} \deg v$$ □

Here's a second proof that formalizes things slightly differently.

*Proof.* Define the set

$$\widehat{E} = \{(v, e) \in V \times E : v \in e\}.$$

We observe two things:

- For any $v \in V$, $|\{e \in E : v \in e\}| = \deg v$ (this is how we defined degree).
- For any $e \in E$, $\{v \in V : v \in e\} = e$, which has size 2.

Therefore

$$|\widehat{E}| = \sum_{v\in V} |\{e \in E : v \in e\}| = \sum_{v\in V} \deg v, \qquad \text{and}$$
$$|\widehat{E}| = \sum_{e\in E} |\{v \in V : v \in e\}| = \sum_{e\in E} 2 = 2|E|,$$

from which the claim follows. □