These notes are from https://mathematicaster.org/teaching/graphs2022/extra_01-20.pdf

Here's an alternative proof of Theorem 1.8 in the book using breaks instead of paths.

Theorem 1. *Let G be a graph on at least three vertices. If there are distinct vertices $u \neq v \in V(G)$ such that both $G-u$ and $G-v$ are connected, then G itself is connected.*

Proof. Consider any partition $V(G) = A \sqcup B$ with A and B nonempty. We need to show that there is some edge of G with one vertex in A and the other in B.

We begin by observing that we can pick $w \in \{u, v\}$ such that $A \neq \{w\}$ and $B \neq \{w\}$. Note that if both A and B have at least two elements, then we can pick w arbitrarily, so, by symmetry, it suffices to consider the case when A is a singleton. First suppose that $A = \{u\}$; then picking $w = v$, we have $A \neq \{w\}$ and also $B \neq \{w\}$ since $|B| \geq 2$. Similarly, if $A = \{v\}$; then picking $w = u$, we have $A \neq \{w\}$ and also $B \neq \{w\}$ since $|B| \geq 2$.

Consider $G - w$, which is connected by assumption. Since $A \neq \{w\}$ and $B \neq \{w\}$, we see that $A \setminus \{w\}$ and $B \setminus \{w\}$ are both nonempty; furthermore, $V(G-w) = (A \setminus \{w\}) \sqcup (B \setminus \{w\})$. Therefore, there must be some $a \in A \setminus \{w\}$ and some $b \in B \setminus \{w\}$ such that $ab \in E(G-w)$. Noting that $a \in A$, $b \in B$ and $ab \in E(G)$ as well concludes the proof □

Here is a slightly different proof of Theorem 1.12 in the book, though it's very similar; I've also included a few more careful details. Recall that a cycle of length n ($n \geq 3$) in a graph G is a sequence of distinct vertices $(v_0, \dots, v_{n-1})$ such that $v_i v_{(i+1) \bmod n} \in E(G)$ for all $i \in \{0, \dots, n-1\}$.

Theorem 2. *G is a bipartite graph if and only if it contains no odd-length cycle.*

Proof. We first observe that G is bipartite if and only if every subgraph of G is bipartite. Indeed, the reverse direction is trivial since G is a subgraph of itself. On the other hand, if $V(G) = A \sqcup B$ is a bipartition of G, and H is any subgraph of G, $A \cap V(H)$ and $B \cap V(H)$ form a bipartition of H (why?).

($\Rightarrow$) We prove the contrapositive. We showed in class that odd-length cycles are not bipartite; therefore, the claim follows from the above observation.

($\Leftarrow$) We again prove the contrapositive. Suppose that G is not bipartite. First, we may assume that G is connected; indeed, if G is not connected then we can break it into connected components $G_1, \dots, G_k$. If each G_i is bipartite, then so is G (why?), so there must be some G_i which is not bipartite: if we find an odd cycle in this G_i, then that odd cycle exists in G as well.

Fix any $v \in V(G)$ and define $N_i = \{u \in V(G) : d(v, u) = i\}$. Note that $N_\infty = \varnothing$ since G is connected, that $N_0 = \{v\}$ and that the N_i's are disjoint.

We begin with an observation: if $xy \in E(G)$ with $x \in N_i$ and $y \in N_j$, then $|i - j| \leq 1$. To prove this, consider a v-x geodesic $(v = v_0, \dots, v_i = x)$ (recall that $d(v, x) = i$). Then since $xy \in E(G)$, we know that $(v = v_0, \dots, v_i = x, y)$ is a v-y walk of length $i + 1$, and so $j = d(v, y) \leq i + 1$. A symmetric argument shows that $i \leq j + 1$ and so the observation holds.

Now define

$$A = \bigcup_i N_{2i}, \qquad B = \bigcup_i N_{2i+1},$$

that is, A is the set of all vertices at an even distance from v and B is the set of all vertices at an odd distance from v. Note that $V(G) = A \sqcup B$ since G is connected. Since G is not bipartite there must be some edge $xy \in E(G)$ completely contained in either A or in B; suppose that $x \in N_i$ and $y \in N_j$. From above, we know that $|i - j| \leq 1$; if $|i - j| = 1$, then i and j have different parity and so $x \in A, y \in B$ or vice versa. Thus, we must have $i = j$, i.e. $x, y \in N_i$; note that $i \geq 1$ since $x \neq y$. Fix v-x v-y geodesics $(v = x_0, \ldots, x_i = x)$ and $(v = y_0, \ldots, y_i = y)$, respectively. Define $j \in \{0, \ldots, i\}$ to be the largest index such that $x_j \in \{y_0, \ldots, y_i\}$; observe the following:

- j exists since $x_0 = y_0$.
- $x_j = y_j$ (and hence $j < i$ since we know $x \neq y$). Indeed, we know that $x_j = y_k$ for some $k \in \{0, \ldots, i\}$. Observe that $(v = x_0, \ldots, x_j = y_k, y_{k+1}, \ldots, y_i = y)$ is a v-y walk of length $j+i-k$, so $j+i-k \geq d(v, y) = i$ implying $j \geq k$. Similarly, $(v = y_0, \ldots, y_k = x_j, \ldots, x_i = x)$ is a v-x walk of length $k + i - j$, so $k + i - j \geq d(v, x) = i$ implying $k \geq j$.

We now consider $(x = x_i, x_{i-1}, \ldots, x_{j+1}, x_j = y_j, y_{j+1}, \ldots, y_{i-1}, y_i = y)$. By the definition of j, all of these vertices are distinct and so this is a cycle in G (since $xy \in E(G)$ by assumption). Finally, we observe that the length of this cycle is $2(i - j) + 1$, which is odd. $\square$