These notes are from https://mathematicaster.org/teaching/graphs2022/extra_01-18.pdf

The book defines connectivity of a graph in terms of walks and paths between the vertices of G. These notes are here to set out another notion of connectivity that can often be useful.

Definition 1. *For vertices $u, v \in V(G)$, a u-v* ***break*** *is a partition $V(G) = A \sqcup B$ such that*

1. *$u \in A$ and $v \in B$, and*
2. *There are no edges between A and B.*

Furthermore, we simply say that a partition $V(G) = A \sqcup B$ with A, B non-empty is a ***break*** *of G if there are no edges between A and B.*

Observe that a partition $V(G) = A \sqcup B$ with A, B non-empty is a break of G if and only if A, B is an a-b break for every $a \in A$ and $b \in B$.

Theorem 2. *For vertices $u, v \in V(G)$, there exists a u-v walk if and only if there is* ***no*** *u-v break.*

Proof. Suppose first that $(u = v_0, v_1, \ldots, v_k = v)$ is a u-v walk. Fix any partition $V(G) = A \sqcup B$ with $u \in A$ and $v \in B$. We must show that there is some edge between A and B. Let $i \in \{0, \ldots, k\}$ be the largest index for which $v_i \in A$. We note two things:

- i exists since $u = v_0 \in A$.
- $i < k$ since $v = v_k \in B$.

Thus, $i + 1 \in \{1, \ldots, k\}$, and, by definition, $v_{i+1} \in B$. We thus see that the edge $v_i v_{i+1}$ has one end-point in A and the other in B.

Suppose next that there is no u-v break; we must find a u-v walk. Let A denote the set of all vertices $w \in V(G)$ such that there is a u-w walk and let B denote the rest of the vertices, so $V(G) = A \sqcup B$. Certainly $u \in A$ since (u) is a u-u walk. If $v \in A$ as well, then we are done, so we may suppose that $v \in B$. Thus, since there is no u-v break, there must be an edge $ab \in E(G)$ with $a \in A$ and $b \in B$. Since $a \in A$, there is a u-a walk, call it $(u = w_0, \ldots, w_k = a)$. But then, $(u = w_0, \ldots, w_k = a, b)$ is a u-b walk, contradicting the fact that $b \in B$. □

Corollary 3. *A graph G is connected if and only if G has no breaks.*

Proof. Consider two relations on $V(G)$:

- $u\, W\, v$ iff there is a u-v walk, and
- $u\, B\, v$ iff there is no u-v break.

Theorem 2 implies that $W = B$. We next observe that B has exactly one equivalence class if and only if G has no breaks. Additionally, G is connected if and only if W has exactly one equivalence class, and so the claim follows. □

Exercise: Consider the relation B on $V(G)$ where $u\, B\, v$ if and only if there is **no** u-v break. Without using the above theorems, show that B is an equivalence relation.