These notes are from https://mathematicaster.org/teaching/graphs2022/dyck.pdf

These are supplementary notes. You will not be tested on these ideas (unless we explicitly cover them later in the semester), but the exercises and ideas contained herein are useful if you want to learn more or want to have more practice problems!

A *rooted tree* is a pair (T, r) where T is a tree and r is some vertex of T. We refer to r as the *root* of the rooted tree (T, r). Note that any tree can be transformed into a rooted tree by simply designating a root-vertex. Since rooted trees are simply trees with a distinguished vertex, we will often denote a rooted tree simply by T and denote the root of T by $\mathrm{root}(T)$.

Recall that if T is a tree, then for every $u, v \in V(T)$, there is a unique u-v path. From this observation, we define a number of relations on a rooted tree.

Definition 1. *Let T be a rooted tree with $r = \mathrm{root}(T)$ and fix vertices $x, y \in V(T)$.*

1. *x is a **descendant** of y if the unique r-x path contains the vertex y.*
2. *y is an **ancestor** of x if x is a descendant of y.*
3. *x is a **child** of y if x is a descendant of y and $xy \in E(T)$. x may also be referred to as a **direct descendant** of y.*
4. *y is the **parent** of x if x is a child of y. y may also be referred to as the **direct ancestor** of x.*
5. *x and y are said to be **siblings** if they have the same parent.*

Problem 1. Let T be a rooted tree. Prove that every $v \in V(T)$ has at most one parent (hence justifying the use of "the" in the definition). Furthermore, prove that the only vertex of T without a parent is $\mathrm{root}(T)$.

Problem 2. Use Problem 1 to show that if T is any tree on n vertices, then T has exactly $n - 1$ edges.

Problem 3. Two rooted trees (T, r) and (T', r') are said to be isomorphic if there is an isomorphism f from T to T' (as a standard graph isomorphism) such that $f(r) = r'$.

Suppose that f is a rooted-tree-isomorphism from (T, r) to (T', r') and fix $x, y \in V(T)$. Prove that x is an descendant/ancestor/child/parent/sibling of y if and only if the same is true of $f(x)$ and $f(y)$.

Definition 2. *Let T be a rooted tree and fix a vertex $v \in V(T)$. The* sub-tree of T rooted at v*, denoted by $T(v)$ is the rooted tree (T', v) where T' is the subgraph of T induced by v and its descendants in T.*

Problem 4. Let T be a rooted tree and fix $v \in V(T)$. Prove that $T(v)$ is indeed a rooted tree. Furthermore, show that all descendant/ancestor/child/parent/sibling relations in $T(v)$ are the same as they were in T.

For a set V, let $\mathcal{T}_V$ denote the set of all trees on vertex-set V. Furthermore, let $\mathcal{RT}_V$ denote the set of all rooted trees on vertex-set V.

In order to go further, we will need to enforce some ordering on the vertex-set of V. When V is an ordered set, $\mathcal{RT}_V$ is known as the set of *rooted plane trees* on vertex-set V. Since these are supplementary notes, I'm honestly too lazy to type up an explanation as to why these are called "plane trees". Come see me in office hours if you want to know more :)

Let V be an ordered set with $|V| = n \geq 1$. We define the following function

$$\mathrm{Dyck}_V : \mathcal{RT}_V \to \{\pm 1\}^{2n-2}$$

recursively as follows:

1. If $n = 1$, then $\mathrm{Dyck}_V(T) = ()$; the empty-sequence.
2. If $n \geq 2$ and $T \in \mathcal{RT}_V$, then suppose that the children of $\mathrm{root}(T)$ are $v_1, \ldots, v_k$ where $v_1 < \cdots < v_k$. Setting $V_i = V\big(T(v_i)\big)$, define

$$\mathrm{Dyck}_V(T) = \big(1, \mathrm{Dyck}_{V_1}(T(v_1)), -1, 1, \mathrm{Dyck}_{V_2}(T(v_2)), -1, \ldots, 1, \mathrm{Dyck}_{V_k}(T(v_k)), -1\big).$$

Technically speaking, $\mathrm{Dyck}_V(T)$ would look something like $(1, (1, (), -1), -1, 1, (), -1)$, but we drop the parentheses and write $(1, 1, -1, -1, 1, -1)$. With this identification, we see that Dyck_V is well-defined.

Since these are supplementary notes, I'm honestly too lazy to type up an explanation as to why this is a natural function to consider. If you're curious about the intuition, come see me in office hours :)

Definition 3. *A sequence $(x_1, \ldots, x_k) \in \{\pm 1\}^k$ is called a **Dyck path** if $\sum_{i=1}^k x_i = 0$ and $\sum_{i=1}^\ell x_i \geq 0$ for all $\ell \in [k]$.*

Note that k must be even in order for a Dyck path of length k to exist.

Problem 5. Let V be a non-empty ordered set. Prove that $\mathrm{Dyck}_V(T)$ is a Dyck path for every $T \in \mathcal{RT}_V$.

Let $\mathrm{DYCK}(n)$ denote the set of all Dyck paths of length n.

Problem 6. Let V be a non-empty ordered set with $|V| = n$. Prove that Dyck_V is a surjection from $\mathcal{RT}_V$ to $\mathrm{DYCK}(2n-2)$.

Unfortunately, Dyck_V is *not* an injection (I encourage you to find an example demonstrating this fact).[1] However, we can understand the equivalence relation generated by Dyck_V.

Definition 4. *Let V be an ordered set and fix $T, S \in \mathcal{RT}_V$. We say that T and S are isomorphic as rooted plane trees if there is a bijection $f : V \to V$ such that*

1. *f is an isomorphism between the rooted trees $(T, \mathrm{root}(T))$ and $(S, \mathrm{root}(S))$, and*
2. *For every $x, y \in V(T)$, if x and y are siblings in T, then $x < y$ if and only if $f(x) < f(y)$.*

Note that we do not require any maintenance of the order if x and y are not siblings. That is to say, if x and y are not *siblings in T, then we could have $x < y$ and $f(x) > f(y)$.*

[1] A word of warning: Some people define rooted plane trees so that Dyck_V is indeed an injection. This is usually done by considering only special rooted plane trees which are "canonical" in some sense. I opted against this in these notes only for notational convenience (so that the set $\mathcal{RT}_V$ does not depend on the order on the set V). But do keep this in mind if you take a general enumerative combinatorics class and your professor discusses this topic.

Note that we could, of course, define isomorphisms between rooted plane trees defined on different ordered sets, but this is unnecessary for our arguments. Observe that if $T, S \in \mathcal{RT}_V$ are isomorphic as rooted plane trees, then $\mathrm{Dyck}_V(T) = \mathrm{Dyck}_V(S)$. Indeed, Dyck_V cares only about the parent-child relation and the ordering among siblings, both of which are unaffected by an isomorphism. The converse is also true:

Problem 7. Let V be a non-empty ordered set. Prove that if $T, S \in \mathcal{RT}_V$ have $\mathrm{Dyck}_V(T) = \mathrm{Dyck}_V(S)$, then T and S are isomorphic as rooted plane trees.

For a non-empty ordered set V, define the equivalence relation $\cong$ on $\mathcal{RT}_V$ where $T \cong S$ iff T and S are isomorphic as rooted planar trees. Let $\widetilde{\mathcal{RT}}_V$ denote the set of equivalence classes of $\cong$. Using the problems above, we have shown that $|\widetilde{\mathcal{RT}}_V| = |\mathrm{DYCK}(2n-2)|$ where $n = |V|$.

Now, Dyck paths are well-studied. If you take a class in general enumerative combinatorics, you will prove that for every non-negative integer n,

$$|\mathrm{DYCK}(2n)| = \frac{1}{n+1}\binom{2n}{n},$$

which are known as the Catalan numbers. I don't expect you to have the tools to prove this in this class. Instead, show the following, much easier, inequality:

Problem 8. $|\mathrm{DYCK}(2n)| \leq \binom{2n}{n} \leq 4^n$ for every non-negative integer n.

We are now ready to finish the point of these notes and get a (still fairly poor) bound on the number of trees on n vertices up to isomorphism. Let V be a non-empty set. Let $\cong$ be the equivalence relation on $\mathcal{T}_V$ where $T \cong S$ iff T and S are isomorphic trees. Let $\widetilde{\mathcal{T}}_V$ denote the set of equivalence classes of $\cong$.

Problem 9. Let V be a non-empty set. Show that if we enforce any arbitrary ordering on V, then $|\widetilde{\mathcal{T}}_V| \leq |\widetilde{\mathcal{RT}}_V|$. Conclude that the number of non-isomorphic trees on n vertices is at most 4^n.

This is still a poor bound for large n. The following is known, though I will not attempt to walk through even an intuition about its proof:

Theorem 5 (Otter's tree enumeration theorem)**.** *When n is very, very large, the number of non-isomorphic trees on n vertices is approximately $\beta n^{-5/2}\alpha^n$ where $\beta \approx 0.5349496$ and $\alpha \approx 2.9557653$.*