I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Let n be any positive integer. Set $N = 2n$ if n is odd and set $N = 2n - 1$ if n is even. Show that every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of $K_{1,n}$.

Problem 2. Show that if G is 3-connected, then any pair of vertices $u \neq v \in V(G)$ are contained together within an even cycle.

Problem 3. Let G be a bipartite graph with parts A, B which has at least one edge and set $D = \{v \in V(G) : \deg v = \Delta(G)\}$. Prove that G contains a matching which saturates $A \cap D$.

(Note: if $A \cap D = \varnothing$, then the "empty matching" suffices)

Problem 4. For an integer $N \geq 2$, let K_N^- denote the graph formed by deleting exactly one edge from K_N (it really doesn't matter which edge since they're all identical).

Fix any integer $n \geq 2$ and set $N = R(n, n)$. By definition, every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of K_n. However, prove that there exists some red,blue-coloring of $E(K_N^-)$ which *does not* contain a monochromatic copy of K_n. In other words, show that every single edge of K_N is important when it comes to forcing a monochromatic copy of K_n.[1]

Problem 5. Recall that the 3-color Ramsey number $R(m, n, p)$ is the smallest integer N such that every 3-coloring of $E(K_N)$ (say with colors red,blue,green) contains either a red K_m, a blue K_n or a green K_p.

1. Show that if $m, n, p \geq 2$, then

$$R(m, n, p) \leq R(m-1, n, p) + R(m, n-1, p) + R(m, n, p-1) - 1.$$

2. Show that if $m, n, p \geq 1$, then

$$R(m, n, p) \leq R\big(m, R(n, p)\big).$$

Can you see how either inequality proves that $R(m, n, p)$ actually exists for all m, n, p? Can you see how to generalize both inequalities to the "t-color Ramsey number"?

Problem 6. Let $C(n)$ denote the set of all red,blue-colorings of $E(K_n)$. Show that

$$\operatorname*{average}_{f \in C(n)} \#\{\text{mono}\chi \text{ triangles in } f\} = \frac{1}{4}\binom{n}{3}.$$

Conclude that there is some red,blue-coloring of $E(K_n)$ in which *strictly* fewer than 1/4 of all triangles are monochromatic.

This shows that the theorem we proved in class is (approximately) tight.

Problem 7.

[1] N.b. If you study more Ramsey theory in the future, you may come across the so-called "size-Ramsey numbers" $\widehat{R}(H)$, which is the fewest number of edges in a graph G such that any 2-coloring of $E(G)$ has a monochromatic copy of the graph H. This problem essentially proves that $\widehat{R}(K_n) = \binom{R(n,n)}{2}$.

1. Show that every red,blue-coloring of $E(K_n)$ must contain a monochromatic tree on n vertices
2. Let T be any tree on t vertices, let n be a positive integer and set $N = n + t - 1$. Prove that any red,blue-coloring of $E(K_N)$ contains either a red copy of T or a blue copy of $K_{1,n}$. (This generalizes part of Problem 1.)
3. Consider any n which is a multiple of 4. Construct a 3-coloring of $E(K_n)$ which *does not* contain a monochromatic tree on strictly more than $\frac{n}{2}$ many vertices.

Problem 8. Show that any t-coloring of $E(K_n)$ contains a monochromatic tree on at least $n/(t-1)$ many vertices.

Roadmap:

1. Start by showing that if G is a bipartite graph with parts A, B, then G contains a tree on at least $\left(\frac{1}{|A|} + \frac{1}{|B|}\right) \cdot |E|$ many vertices. In particular, show it contains a "double star" of this size, where a double star is two stars whose centers are connected by an edge.
   (a) Use Cauchy–Schwarz[2] and the bipartite handshaking lemma to show that
   $$\sum_{\substack{a\in A, b\in B:\\ ab\in E}} (\deg a + \deg b) \geq \left(\frac{1}{|A|} + \frac{1}{|B|}\right) \cdot |E|^2.$$
   (b) Why does this give you the desired double star?
2. Now consider a t-coloring of $E(K_n)$. If the color-t-graph is connected, then we win. Otherwise, the color-t-graph has a break; apply part 1 to this break in some way.

Problem 9. In class, we showed that any sequence of n *distinct* real numbers contains a monotone subsequence of length $> \frac{1}{2}\log_2 n$. Use DS5.3.4 to improve this bound to $\geq \sqrt{n}$ (which is actually the correct answer).

Problem 10. You just got a new TV, but the remote didn't come with any batteries and requires two batteries to operate... Luckily, you have a box of $2n \geq 4$ old batteries to fuel your new remote! You remember that exactly n of these batteries are completely dead and exactly n of these batteries have at least some charge. Unfortunately, they are scattered about and you can't tell which is which without testing them in your new remote. So, all you can do is insert two of these batteries into your remote and see if it works. The remote will not work at all if even one of the inserted batteries is dead. Determine (in terms of n) the fewest number of trials needed to make your remote work.

(To help you along, the correct answer is 6 if $n = 2$ and is $n + 3$ if $n \geq 3$.)

Problem 11. Let T be any tree on $t \geq 2$ vertices.

1. Prove that $\mathrm{ex}(n, T) \geq \frac{1}{2}(t-2)n$ whenever $(t-1) \mid n$.
2. Prove that $\mathrm{ex}(n, T) \leq (t-2)n$.

   (Hint: Show that any graph G contains a subgraph H with $\delta(H) \geq |E(G)|/|V(G)|$. To show this, consider taking H to be a subgraph of G which maximizes the quantity $|E(H)|/|V(H)|$.)

N.b. A conjecture of Erdős and Sós from the 60's posits that $\mathrm{ex}(n, T) \approx \frac{1}{2}(t-2)n$ where the "$\approx$" is simply a rounding error if $(t-1) \nmid n$.

[2] in particular, the special case which stated: $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}\left(\sum_{i=1}^n a_i\right)^2$