This worksheet is from https://mathematicaster.org/teaching/graphs2022/ds4.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. You and your friends want to tour the American southwest by car (see the picture below). For some strange reason (probably since you're all nerds and are taking this class :)), you decide you wish to cross the border between every pair of neighboring states[1] exactly once (and not visit any states not in the southwest). Can you accomplish this peculiar goal? If so, does it matter where you start/end your road trip? Justify your answer.

Problem 2. Let G be a bipartite graph with parts A, B where $|A| \geq |B|$. Of course, $\alpha'(G) \leq |B|$. Furthermore, since A is an independent set, certainly we have $\alpha(G) \geq |A|$.

Prove that $\alpha(G) = |A|$ if and only if $\alpha'(G) = |B|$.

Problem 3. Let $G = (V, E)$ be a graph. We say that two non-empty sets $A, B \subseteq V$ (which could intersect) form a *disjunction* of G if G has no edge with one end-point in A and the other in B. Note that disjunctions always exist since we could take $A = B = \{v\}$ for some vertex v.

Prove that for any graph G (yes, even cliques),

$$\kappa(G) = |V| - \max_{A,B \text{ a disjunction of } G} |A \cup B|.$$

Problem 4. We have a standard deck of 52 cards, so there are 4 suits and 13 ranks. We deal the cards (arbitrarily) into 13 piles, each containing four cards. Prove that we can select exactly one card from each pile so that we wind up with each rank.

Problem 5. Show that any connected graph contains a walk which traverses each edge *exactly* twice.

(Extra fun: Prove this both with and without invoking multigraphs)

Problem 6. Fill in the "BLAH" in the following statement (and prove it): If G is a connected graph, then G contains a walk which traverses each edge *exactly* thrice if and only if BLAH.

(Hint: You probably want to use multigraphs here even though G is simple)

Problem 7. There are a bunch of islands connected by a bunch of bridges. You are given a torch and charged with burning all of these bridges. Of course, you can only burn bridges which touch the island you're currently standing on and you can't cross a bridge that you've previously burned. Furthermore, if you ever cross a bridge, you must burn that bridge behind you.

[1] We do not consider Utah and New Mexico nor Colorado and Arizona to be neighboring states since they touch only at a point.

Using objects/terms that we've been discussing lately, determine exactly when you can burn all of the bridges assuming that you get to decide on which island you begin your rampage.

(Note: I don't know if there are reasonable necessary/sufficient conditions on, say, the degrees of this island-bridge network which work, though I doubt it.)

Problem 8. A *Latin square of order* n is a matrix $A \in [n]^{n\times n}$ such that every $i \in [n]$ appears exactly once in each row and each column of A. Note that any Sudoku is a Latin square of order 9, although there are additional rules in Sudoku.

Suppose that $B \in [n]^{m\times n}$ for some $m < n$ and each $i \in [n]$ appears exactly once in each row of B and at most once in each column of B. Prove that B can be extended to a Latin square of order n (i.e. one can fill in the remaining $n - m$ rows of B to create a valid Latin square).

(Hint: Just show that one can fill in the $(m+1)$st row and then conclude by induction on $n-m$)

(Hint: Relate the problem of filling in the $(m + 1)$st row to one of finding a system of distinct representatives (or directly to finding a matching in a bipartite graph))

Problem 9. Suppose that G is a graph with no isolated vertices.

1. Show that if G is connected and has no copy of P_4, then either $G \cong K_3$ or $G \cong K_{1,n}$ for some $n \geq 1$.

2. Suppose that $S \subseteq E(G)$ is a minimum edge-cover of G, so $|S| = \beta'(G)$. Prove that (V, S) is a forest with no isolated vertices wherein each connected component is a star (i.e. a copy of $K_{1,n}$ for some $n \geq 1$).

Problem 10. Let $G = (V, E)$ be a graph. For a function $f : E \to \{0, 1\}$, $i \in \{0, 1\}$ and $v \in V$, we define

$$\deg_i^f v = |\{e \in E : v \in e \text{ and } f(e) = i\}|.$$

Observe that $\deg_0^f v + \deg_1^f v = \deg v$ for any $v \in V$.

1. Suppose that G is an even-regular graph. Show that if $|E|$ is odd and $f : E \to \{0, 1\}$ is any function, then there must be some $v \in V$ such that $|\deg_0^f v - \deg_1^f v| \geq 2$.

2. Suppose that G is a connected, even-regular graph and fix any $v^* \in V$. Prove that there is a function $f : E \to \{0, 1\}$ satisfying

   (a) $\deg_0^f v = \deg_1^f v$ for all $v \in V \setminus \{v^*\}$, and

   (b) If $|E|$ is even, then also $\deg_0^f v^* = \deg_1^f v^*$, and

   (c) If $|E|$ is odd, then $|\deg_0^f v^* - \deg_1^f v^*| = 2$.

3. Suppose that G is a connected graph with at least one odd-degree vertex. Prove that there is a function $f : E \to \{0, 1\}$ such that $|\deg_0^f v - \deg_1^f v| \leq 1$ for all $v \in V$.

N.b. It turns out that the sorta converse holds: If G is connected and $f : E \to \{0, 1\}$ is a function such that $\deg_0^f v = \deg_1^f v$ for all $v \in V$, then G has an Eulerian circuit $(v_0, e_1, \ldots, v_{m-1}, e_m, v_m = v_0)$ such that $f(e_i) = i \mod 2$ for all $i \in [m]$. Proving this fact requires proving analogues of all of the lemmas needed to establish the existence of Eulerian circuits. This is a good exercise if you care to put in the work.