This worksheet is from https://mathematicaster.org/teaching/graphs2022/ds2.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Show that every tree is bipartite.

Problem 2. Show that if G has $|E(G)| = |V(G)| + k$ for some integer $k \geq -1$, then G contains at least $k+1$ distinct cycles (though these cycles may overlap substantially).

Problem 3 (Fulkerson–Hoffman–McAndrew conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^k d_i \leq k(m-1) + \sum_{i=m+1}^n \min\{d_i, k\}, \qquad \text{for every } k, m \in [n] \text{ with } k \leq m.$$

Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Problem 4 (Bollobás conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^k d_i \leq \sum_{i=1}^k \min\{d_i, k-1\} + \sum_{i=k+1}^n \min\{d_i, k\}, \qquad \text{for every } k \in [n].$$

Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Problem 5 (Grünbaum conditions)**.** Prove that if $d_1 \geq \cdots \geq d_n$ is graphical, then $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^k \max\{d_i, k-1\} \leq k(k-1) + \sum_{i=k+1}^n \min\{d_i, k\}, \qquad \text{for every } k \in [n].$$

Note that these conditions imply the Erdős–Gallai conditions (why?) and so this is actually a biconditional statement.

Problem 6 (Ryser conditions)**.** A pair of sequences $(a_1, \ldots, a_m), (b_1, \ldots, b_n)$ is said to be *bipartite-graphical* if there is a bipartite graph G with parts $A = \{v_1, \ldots, v_m\}$ and $B = \{u_1, \ldots, u_n\}$ such that $\deg v_i = a_i$ for all $i \in [m]$ and $\deg u_i = b_i$ for all $i \in [n]$.

Suppose that $d_1, \ldots, d_n$ is a graphical sequence. Show that if $\gamma_1, \ldots, \gamma_n$ is any sequence such that $\gamma_i \in \{d_i, d_i + 1\}$ for all $i \in [n]$, then the pair of sequences $(\gamma_1, \ldots, \gamma_n), (\gamma_1, \ldots, \gamma_n)$ is bipartite-graphical.

Turns out that this is actually a biconditional statement if we include the condition that $\sum_{i=1}^n d_i$ is even. That is to say that if $\sum_{i=1}^n d_i$ is even and the pair $(\gamma_1, \ldots, \gamma_n), (\gamma_1, \ldots, \gamma_n)$ is bipartite-graphical for all sequences $\gamma_1, \ldots, \gamma_n$ with $\gamma_i \in \{d_i, d_i + 1\}$ for all i, then $d_1, \ldots, d_n$ is graphical. (I don't expect you to prove this.)

Problem 7. This problem will walk through another proof that trees on n vertices have $n-1$ edges. Let T be a tree and fix any vertex $u \in V(T)$. For each non-negative integer i, set $N_i = \{v \in V(T) : d(u, v) = i\}$. Note that $N_0 = \{u\}$ and that N_1 is the neighborhood of u.

1. Show that $V(T) = \bigsqcup_{i \geq 0} N_i$.
2. Show that if $xy \in E(T)$, then there is some $i \geq 0$ such that $x \in N_i$ and $y \in N_{i+1}$ (or vice versa).
3. Show that for each $i \geq 1$ and any $v \in N_i$, v has exactly one neighbor in N_{i-1}.
4. Use these facts to prove that $|E(T)| = |V(T)| - 1$.

Problem 8. This problem shows that the main idea in problem 7 actually classifies all trees. Let G be a graph and suppose that there is a partition $V(G) = V_0 \sqcup \cdots \sqcup V_k$ with the following properties:

1. V_i is non-empty for all $i \in \{0, \ldots, k\}$, and
2. $G[V_0]$ is a tree, and
3. V_i is an independent set for all $i \in [k]$, and
4. For each $i \in [k]$ and any $v \in V_i$, v has exactly one neighbor in $\bigcup_{j=0}^{i-1} V_j$.

Show that G is a tree.

Problem 9 (Borůvka's algorithm)**.** Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function which assigns distinct weights (i.e. $w(e) \neq w(s)$ for any distinct $e, s \in E(G)$).

Initialize $F = (V(G), \varnothing)$ and iterate the following process:

1. If F is connected, terminate and return F.
2. If F is disconnected, suppose that $F_1, \ldots, F_k$ are the connected components of F. For each $i \in [k]$, let e_i be the minimum weight edge which has exactly one vertex in F_i (note: there cannot be any ties since w assigns distinct weights). Replace F by $F + e_1 + \cdots + e_k$ and repeat (note that it's possible that some of the e_i's are the same — if this happens, we add that edge in only once).

Prove that this algorithm returns a minimum weight spanning tree of G. (Note: The most difficult part is arguing that the graph returned is acyclic.)

Extra fun: Why did we need to require that w assigned distinct weights? How could you modify Borůvka's algorithm if this is not the case?

Extra extra fun: Show that Borůvka's algorithm terminates after at most $\lfloor \log_2 |V(G)| \rfloor$ iterations.

Problem 10 (Challenge question ☺)**.** Show that if $|E(G)| \geq 2|V(G)|$, then G contains a cycle of length at most $2\lceil \log_2 |V(G)| \rceil - 1$.