This worksheet is from https://mathematicaster.org/teaching/graphs2022/ds1.pdf

I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. State three equivalent definitions of graph connectivity.

Problem 2. Show that any minim**al** u-v walk is a u-v path.

Problem 3. Prove that for any $u, v, w \in V(G)$, $d(u, w) \leq d(u, v) + d(v, w)$. (Pay careful attention if G happens to be disconnected!)

Problem 4. Let G be a graph. Define the relation R on $V(G)$ where $u \, R \, v$ if and only if for every partition $V(G) = A \sqcup B$ with $u \in A$ and $v \in B$, there is at least one edge between A and B. Prove that R is an equivalence relation on $V(G)$.

Problem 5. Let G be a connected graph Suppose that there are no three distinct vertices $u, v, w \in V(G)$ such that $uv, uw \in E(G)$ yet $vw \notin E(G)$. Prove that if G has n vertices, then $G \cong K_n$.

Problem 6. Prove the "handshaking dilemma": If $D = (V, E)$ is a digraph, then

$$\sum_{v \in V} \deg^+ v = \sum_{v \in V} \deg^- v = |E|.$$

Here, $\deg^+ v = |\{u \in V : (v, u) \in E\}|$ and $\deg^- v = |\{u \in V : (u, v) \in E\}|$ denote the out- and in-degree of v, respectively.

Problem 7. Fix a graph $G = (V, E)$. The *line graph* of G, denoted $L(G)$, has vertex set E and edge set $\left\{es \in \binom{E}{2} : e \cap s \neq \varnothing\right\}$.

1. Prove that $L(C_n) \cong C_n$ for every $n \geq 3$.
2. Prove that $L(P_n) \cong P_{n-1}$ for every $n \geq 2$.
3. For $uv \in E$, prove that $\deg_{L(G)}(uv) = \deg_G(u) + \deg_G(v) - 2$.
4. Prove that $L(G)$ is connected whenever G is connected.
5. If $L(G)$ is connected, must it be the case that G is connected?

Problem 8. Consider a digraph $D = (V, E)$. Recall that a u-v diwalk is a sequence of vertices $(u = v_0, \ldots, v_k = v)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, k-1\}$. D is said to be *strongly connected* if for every $u, v \in V$, there is both a u-v diwalk and a v-u diwalk (consult HW2.1 for why this definition is reasonable).

Prove that D is strongly connected if and only if for any partition $V = A \sqcup B$ with A, B nonempty, there is $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $(a_1, b_1), (b_2, a_2) \in E$ (note that we could have $a_1 = a_2$ and/or $b_1 = b_2$).

Problem 9. Show that there is no graph G with the property that $\deg_G v \equiv \deg_{\overline{G}} v \equiv 1 \pmod 2$ for every $v \in V(G)$.

Problem 10. Prove that $G \cong H$ if and only if $\overline{G} \cong \overline{H}$.

Problem 11. Let G be a bipartite graph with parts A, B. Show that

$$\sum_{a \in A} \deg a = \sum_{b \in B} \deg b.$$

Problem 12. A *set system* is a pair (V, E) where V is a set and $E \subseteq 2^V$ (i.e. a collection of subsets of V). (Note that a graph is simply a set system wherein each set has size 2). For a set-system (V, E), we can define the degree of a vertex $v \in V$ to be $\deg v = |\{e \in E : v \in e\}|$, exactly the same as we did with graphs.

Use problem 11 to prove the generalization of the handshaking lemma to a set system (V, E):

$$\sum_{v \in V} \deg v = \sum_{e \in E} |e|.$$

(Observe how this implies the original handshaking lemma).

Problem 13. A group of sociologists are curious about the affect of eye color on childhood friendship. So they gather a group of 100 children, each having either brown or blue eyes, and observe the children's play over the course of a day. After gathering the data, the researchers are surprised to find that the average blue-eyed child played with 3 times as many brown-eyed children as the number of blue-eyed children the average brown-eyed child played with. Based on this disparity, the sociologists conclude that blue-eyed children are significantly more likely to look past physical differences and make friends.

Later, you (a student in this graph theory class) have dinner with your friend who was part of this group of sociologists, and they explain the experiment and their excitement about the results. In confusion, you reply, "Why are you so excited? Did you not know there were 75 brown-eyed children and 25 blue-eyed children in your study?"

Explain your response.