These notes are from https://mathematicaster.org/teaching/graphs2022/dilworth.pdf

We proved two fundamental theorems about matchings in bipartite graphs:

Theorem 1 (Kőnig's Theorem)**.** *If G is a bipartite graph, then $\alpha'(G) = \beta(G)$.*

Theorem 2 (Hall's Theorem)**.** *If G is a bipartite graph with parts A, B, then G has a matching which saturates A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.*

We showed that Kőnig $\Longrightarrow$ Hall and Hall $\Longrightarrow$ Kőnig, so, morally speaking, these are the same theorem.[1]

These notes exist to discuss one more theorem which is equivalent to Kőnig and Hall that, at a first glance, seems to have nothing to do with graphs whatsoever. Instead, it is a statement about partially-ordered sets.

Definition 3. *Let P be a non-empty set. A* partial-order *on P is a relation $\preceq$ which satisfies three properties:*

- *Reflexivity: $x \preceq x$ for all $x \in P$.*
- *Anti-symmetry: $x \preceq y \ \wedge \ y \preceq x \implies x = y$ for all $x, y \in P$.*
- *Transitivity: $x \preceq y \ \wedge \ y \preceq z \implies x \preceq z$ for all $x, y, z \in P$.*

So, partial-orders are similar to equivalence relations except we replace symmetry by anti-symmetry.[2]

A pair $(P, \preceq)$ where $\preceq$ is a partial-order on P is known as a *partially-ordered set* or a *poset*. When the partial-order is understood, we simply write P in place of $(P, \preceq)$. In general, for any non-empty set P, there are tons of partial-orders on P; usually we just care about those partial-orders which "make sense" given the nature of the set P.

Note that $\mathbb{R}$ with the usual notion of $\leq$ is a poset. Another common example is the set of all subsets of some fixed set, which is a poset under the partial-order $\subseteq$. Other examples include $\mathbb{N}$ with the relation $x \preceq y$ iff $x \mid y$. In general, for any non-empty set P, we can build the "trivial" poset $(P, \preceq)$ where $x \preceq y \iff x = y$, but this is generally far from interesting.

Definition 4. *Let $(P, \preceq)$ be a poset.*

- *For $x, y \in P$, we say that x and y are* comparable *if $x \preceq y$ or $y \preceq x$. Otherwise we say that x and y are* incomparable*. Note that x is always comparable to itself.*
- *A subset $C \subseteq P$ is called a* chain *if every pair of elements of C are comparable.*

[1] Any logicians in the audience will (correctly) reply "Duh; all true theorems are equivalent". This is certainly true... I mean that each can be "directly" derived from the other (hence my use of the term "morally speaking"). This is definitely not a rigorous statement (nor am I aware of any way to make such a statement rigorous), but hopefully you get the point.

[2] This is completely besides the point, but it's a dumb joke that I like to make. Love fails all three conditions to be an equivalence relation: you may not love yourself; if you love someone, then they may very well not love you back; if you love someone and they love someone else, then there's a good chance that don't love that someone else! Furthermore, we better damn well hope that love is not a partial order! Indeed, if love were a partial order, then if you love someone (other than yourself), then they will never love you back...

- *A subset $A \subseteq P$ is called an* anti-chain *if every pair of distinct elements of A are incomparable.*

Note that a single-element subset is both a chain and an anti-chain.

Definition 5. *Let P be a poset.*

- *The* height *of P, denoted by $h(P)$, is the size of a largest chain in P.*
- *The* width *of P, denoted by $w(P)$, is the size of a largest anti-chain in P.*
- *A* chain-cover *of P is a partition $P = C_1 \sqcup \cdots \sqcup C_k$ where each C_i is a chain in P. The* chain-cover-number *of P, denoted by $c(P)$, is the smallest k for which there is a chain-cover using k chains.*
- *An* anti-chain-cover *of P is a partition $P = A_1 \sqcup \cdots \sqcup A_k$ where each A_i is an anti-chain in P. The* anti-chain-cover-number *of P, denoted by $a(P)$, is the smallest k for which there is an anti-chain-cover using k anti-chains.*

As a quick example, if $P = (2^{[n]}, \subseteq)$, then $h(P) = n + 1$ as witnessed by the chain $\{\varnothing, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}\}$ (why can't there be a larger chain?). Furthermore, $w(P) = \binom{n}{\lfloor n/2 \rfloor}$ as witnessed by the anti-chain $\binom{[n]}{\lfloor n/2 \rfloor}$ (it is non-trivial that this is actually the largest anti-chain; this result is known as Sperner's theorem[3]).

Dilworth's Theorem relates these four parameters.

We begin with a simple observation:

Lemma 6. *Let P be a poset. If C is a chain in P and A is an anti-chain in P, then $|A \cap C| \leq 1$.*

Proof. If $|A \cap C| \geq 2$, then we could find $x \neq y \in A \cap C$. Since $x, y \in C$, we know that x and y are comparable. But then x and y would be distinct elements of A which are comparable, so A cannot be an anti-chain. □

From here, we can start to relate the four parameters of a poset.

Lemma 7. *Let P be a finite poset. Then $h(P) \leq a(P)$ and $w(P) \leq c(P)$.*

Proof. We prove first that $h(P) \leq a(P)$. Let C be a largest chain in P (so $|C| = h(P)$) and let $P = A_1 \sqcup \cdots \sqcup A_k$ be a smallest anti-chain-cover of P (so $k = a(P)$). Then Lemma 6 implies that

$$h(P) = |C| = \sum_{i=1}^{k} |C \cap A_i| \leq \sum_{i=1}^{k} 1 = k = a(P).$$

The proof that $w(P) \leq c(P)$ is similar. Let A be a largest anti-chain in P (so $|A| = w(P)$) and let $P = C_1 \sqcup \cdots \sqcup C_k$ be a smallest chain-cover of P (so $k = c(P)$). Then Lemma 6 implies that

$$w(P) = |A| = \sum_{i=1}^{k} A \cap C_i \leq \sum_{i=1}^{k} 1 = k = c(P).$$ □

The cool thing is that $h(P) = a(P)$ and $w(P) = c(P)$ for any finite poset P, which is Dilworth's theorem. Well, really Dilworth's theorem is just that $w(P) = c(P)$. This is because the fact that $h(P) = a(P)$ is "easy"; hence, I'll call this result "Dumb-Dilworth". Let's prove it quickly!

[3]There are many proofs of Sperner's theorem, the best of which (IMHO) invoke random variables in some way. However, one can also use HW10.4 along with some extra effort to prove this. I'm not 100% sure, but I believe Sperner's original proof followed along these lines laid out in HW10.4 in order to create a notion of "compression".

Theorem 8 (Dumb-Dilworth)**.** *If $(P, \preceq)$ is a finite poset, then $h(P) = a(P)$.*

Proof. We already showed that $h(P) \leq a(P)$, so we just need to show that $a(P) \leq h(P)$. We do so by constructing an anti-chain cover of P using at most $h(P)$ many anti-chains.

Fix any $x \in P$ and a chain $C \subseteq P$. We say that the chain C *ends at* x if

- $x \in C$, and
- For any $y \in C$, $y \preceq x$.

Let $h(x)$ denote the size of a largest chain which ends at x. We claim that $h(x) \in [h(P)]$ for all $x \in P$. Indeed, we know that $h(x) \geq 1$ since $\{x\}$ is a chain which ends at x. Additionally, we know that $h(x) \leq h(P)$ since $h(P)$ is the size of a largest chain in P.

For each $i \in [h(P)]$, we define $A_i = \{x \in P : h(x) = i\}$.[4] By the previous observation, we know that

$$P = A_1 \sqcup \cdots \sqcup A_{h(P)},$$

thus we will have proved the claim if we can show that each A_i is an anti-chain.

Suppose for the sake of contradiction that A_i is *not* an anti-chain; thus there is some $x \neq y \in A_i$ that are comparable. Without loss of generality, $x \preceq y$.

Let C be a largest chain which ends at x, so $|C| = h(x)$. If $y \in C$, then $y \preceq x$ and so $x = y$ due to anti-symmetry; a contradiction. Thus, consider $C' = C \cup \{y\}$, which has size $h(x) + 1$. We claim that C' is a chain which ends at y. Indeed, $y \in C'$ by definition. Also, if $z \in C'$, then

- $z = y$, in which case $z \preceq y$ due to reflexivity, or
- $z \neq y$, in which case $z \in C$. Then, since C is a chain which ends at x, we have $z \preceq x$. Finally, since $x \preceq y$ by assumption, due to transitivity, we must have $z \preceq y$.

Thus, C' is a chain which ends at y and so $h(y) \geq |C'| = h(x) + 1 > h(x)$, which contradicts the fact that $h(x) = h(y) = i$. □

Now we're ready for Dilworth's actual theorem (note: this is highly non-trivial).

Theorem 9 (Dilworth's Theorem)**.** *If $(P, \preceq)$ is a finite poset, then $w(P) = c(P)$.*

Proof. We have already shown that $w(P) \leq c(P)$, so we just need to show that $c(P) \leq w(P)$. We do so by constructing a chain-cover of P using at most $w(P)$ many chains. This is accomplished via Kőnig's theorem.

We start with two "copies" of the elements of P:

$$X = \{x^- : x \in P\}, \quad \text{and} \quad Y = \{x^+ : x \in P\}.$$

Note that the $\pm$ is just notation to distinguish the elements of X and Y. We build a bipartite graph G which has parts X and Y where $x^-y^+ \in E(G)$ if and only if $x \prec y$ (where $x \prec y$ means $x \preceq y$ and $x \neq y$).

Let $R \subseteq V(G)$ be a minimum vertex-cover of G (so $|R| = \beta(G)$). We use R to build an anti-chain in P.

Claim 10. *For every $x \in P$, R contains at most one of x^- and x^+.*

[4] If you know a bit about posets, the A_i's are the level-sets of P.

Proof. Suppose that x^- and x^+ are both members of R. Since R is a minimum vertex-cover of G, we know that $R \setminus \{x^-\}$ cannot be a vertex-cover of G. In particular, there must be some edge $x^-y^+ \in E(G)$ where $y^+ \notin R$. Similarly, $R \setminus \{x^+\}$ cannot be a vertex-cover of G, so there must be some edge $z^-x^+ \in E(G)$ where $z^- \notin R$. By definition, we have $z \preceq x$ and $x \preceq y$, so $z \preceq y$ due to transitivity. Since edges of the form w^-w^+ never exist in G, we actually must have $y \neq z$ since otherwise anti-symmetry would imply that $x = y = z$.

This, however, means that $z \prec y$ and so $z^-y^+ \in E(G)$. But, neither z^- nor y^+ are elements of R, so R does not cover the edge z^-y^+; a contradiction. □

Now, define $A = \{x \in P : R \cap \{x^-, x^+\} = \varnothing\}$.

Claim 11. *A is an anti-chain in P and $|A| = |P| - \beta(G)$.*

Proof. Suppose that A were not an anti-chain; then we could locate $x \neq y \in A$ such that $x \prec y$. But then $x^-y^+ \in E(G)$. Since R is a vertex-cover, either $x^- \in R$ or $y^+ \in R$ (or both), so actually either $x \notin A$ or $y \notin A$ by definition; a contradiction.

To conclude the claim, we must show that $|A| = |P| - \beta(G)$. Now, $|R| = \beta(G)$ and, thanks to Claim 10, at most one of x^- and x^+ live in R. Thus, $|R| = |\{x \in P : R \cap \{x^-, x^+\} \neq \varnothing\}|$, so $|A| + |R| = |P|$, which concludes the proof. □

With the help of the above claims, we have constructed an anti-chain in P of size $|P| - \beta(G)$, and so $w(P) \geq |P| - \beta(G)$.

Now, let $M \subseteq E(G)$ be a maximum matching, so $|M| = \alpha'(G)$. We use M to build a chain-cover of P.

Define

$$U^- = \{x \in P : x^- \text{ is covered by } M\}, \quad U^+ = \{x \in P : x^+ \text{ is covered by } M\}, \quad U = U^- \cup U^+.$$

Note that U^- and U^+ could intersect.

We build a digraph D which has vertex set P and $(x, y) \in E(G)$ iff $x^-y^+ \in M$. Since M is a matching, we find that:

- If $x \in U^- \setminus U^+$, then $\deg^+ x = 1$ and $\deg^- x = 0$.
- If $x \in U^+ \setminus U^-$, then $\deg^- x = 1$ and $\deg^+ x = 0$.
- If $x \in U^- \cap U^+$, then $\deg^\pm x = 1$.

This implies that D is a disjoint union of directed paths, each of length at least one (why?). Furthermore, if $(x_0, \ldots, x_k)$ is one of these directed paths, then $x_0 \in U^- \setminus U^+$, $x_k \in U^+ \setminus U^-$ and $x_1, \ldots, x_{k-1} \in U^- \cap U^+$. In particular, D is the disjoint union of $|U^- \setminus U^+| = |U^+ \setminus U^-|$ many directed paths. Now, if $(x_0, \ldots, x_k)$ is one of these directed paths, then $x_i^- x_{i+1}^+ \in M$ for each $i \in \{0, \ldots, k-1\}$. Thus, $x_0 \preceq \cdots \preceq x_k$ and so $\{x_0, \ldots, x_k\}$ is a chain in P. Using these chains, we have shown that we can cover $U \subseteq P$ by at most $|U^- \setminus U^+| = |U^+ \setminus U^-|$ many chains. We can then cover all of $P \setminus U$ with $|P \setminus U|$ many chains by simply taking each element to be its own chain. Thus, we have constructed a chain-cover of P using at most

$$|U^- \setminus U^+| + |P \setminus U| = P + |U^- \setminus U^+| - |U| = |P| - |U^+|$$

many chains. Certainly $|U^+| = |M| = \alpha'(G)$, so we have shown that $c(P) \leq |P| - \alpha'(G)$.

Finally, Kőnig tells us that $\alpha'(G) = \beta(G)$ and so

$$c(P) \le |P| - \alpha'(G) = |P| - \beta(G) \le w(P). \qquad \square$$

So, we just used Kőnig to prove Dilworth. Now let's do the reverse, thus showing that Kőnig, Hall and Dilworth are all morally the same theorem, despite their apparent differences.

Dilworth $\Longrightarrow$ Kőnig. Let G be a bipartite graph with parts X, Y; we need to prove that $\alpha'(G) = \beta(G)$. Recall that $\alpha'(G) \le \beta(G)$ always, so we just need to prove that $\beta(G) \le \alpha'(G)$.

We create a poset $(P, \preceq)$ from G where $P = V(G)$ and $x \preceq y$ iff $x \in X$, $y \in Y$ and $xy \in E(G)$.[5] Note that $h(P) \le 2$.

Let $P = C_1 \sqcup \cdots \sqcup C_k$ be a minimum chain cover of P, so $k = c(P)$. We can label these chains so that $|C_1| \ge \cdots \ge |C_k|$ Now, each chain has at least one element and at most two elements (since $h(P) \le 2$). Thus, let $\ell \in \{0, \ldots, k\}$ be such that $|C_1| = \cdots = |C_\ell| = 2$ and $|C_{\ell+1}| = \cdots = |C_k| = 1$. In particular, $|P| = 2\ell + (k - \ell) = k + \ell$. Now, for each $i \in [\ell]$, $C_i = \{x, y\}$ where $x \in X$, $y \in Y$ and $xy \in E(G)$; i.e. C_i corresponds to an edge of G. Since $C_1, \ldots, C_\ell$ are disjoint, these ℓ corresponding edges are vertex disjoint and hence a matching. Thus, $\alpha'(G) \ge \ell$. In other words, $|P| = k + \ell \le c(P) + \alpha'(G) \implies \alpha'(G) \ge |P| - c(P)$.

Now, let $A \subseteq P$ be a maximum anti-chain, so $|A| = w(P)$. Then consider $B = P \setminus A = V(G) \setminus A$. We claim that B is a vertex-cover of G. Indeed, if there were some uncovered $xy \in E(G)$ ($x \in X, y \in Y$), then we would have $x, y \in A$. But $x \prec y$, so this is impossible since A is an antichain. Thus, $\beta(G) \le |B| = |P| - |A| = |P| - w(P)$.

Finally, Dilworth tells us that $w(P) = c(P)$ and so

$$\beta(G) \le |P| - w(P) = |P| - c(P) \le \alpha'(G). \qquad \square$$

[5]If you already know a bit about posets, we're creating P so that G is the Hasse diagram of P.