This document is from https://mathematicaster.org/teaching/discrete2026/ws/ws5.pdf

Problem 1. Use a sign-changing involution to prove that the number of surjections from $[m]$ to $[n]$ is

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m.$$

Hint: Surjections correspond to words in which each letter appears at least once. Consider the set $\{(X,Y) : X \subseteq [n],\ Y \in ([n] \setminus X)^m\}$.

Problem 2. Let D_n denote the number of derangements of $[n]$. We have previously shown that $D_n \sim \frac{n!}{e}$. In this problem, we will show that D_n is actually $\frac{n!}{e}$ rounded to the nearest integer.

- Show that for every $n \geq 1$,
$$0 < \sum_{k \geq n} \frac{(-1)^{k-n}}{k!} < \frac{1}{n!}.$$
  Hint: Group together even and odd terms.

- Show that
$$\left|\frac{n!}{e} - D_n\right| < \frac{1}{2}.$$