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Problem 1. Let π be a permutation chosen uniformly at random from S_n.

- Let $C_k(\pi)$ denote the number of k-cycles in π (recall that a cycle is a set of points such that $x_1 \to x_2 \to \cdots \to x_k \to x_1$ under π). Show that $\mathbb{E}\, C_k = \frac{1}{k}$.
- Use Markov's inequality to find an upper-bound on the probability that π has a cycle of length at least $n/2$ as $n \to \infty$.
- A malicious prison warden plays a game with n prisoners. Uniformly at random, pictures of the prisoners distributed among n boxes. One-by-one, the prisoners are allowed to open any $n/2$ boxes they desire. If they find their own picture, then they succeed and otherwise they fail. If all prisoners succeed, then they're all released and otherwise they're all executed! The prisoners are allowed to discuss a strategy before the pictures are distributed, but no communication is allowed afterward.

  Devise some strategy that guarantees the probability of release is at least $\approx 30\%$ for large n.

  N.b. It turns out that this is optimal: www.cl.cam.ac.uk/~gw104/Locker_Puzzle.pdf.

Problem 2. Consider uniformly at random distributing n balls into n bins. Let L denote the size of the largest bin. We will show that L tends to be fairly small.

For a set of k balls $B \in \binom{[n]}{k}$, let X_B be the random variable which returns 1 when all balls in B are put into the same bin and 0 otherwise. Set $X = \sum_{B \in \binom{[n]}{k}} X_B$.

- Show that $X \geq 1$ if and only if $L \geq k$.
- Prove that
$$\mathbb{E}\, X = \frac{1}{n^{k-1}} \binom{n}{k}$$
- Show that for every $\epsilon > 0$,
$$\mathbf{Pr}\left[L \geq (1+\epsilon) \frac{\log n}{\log \log n} \right] \to 0 \qquad \text{as } n \to \infty.$$

  N.b. This is tight in the sense that
$$\mathbf{Pr}\left[L \leq (1-\epsilon) \frac{\log n}{\log \log n} \right] \to 0 \qquad \text{as } n \to \infty.$$