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Problem 1. The variance of a random variable X is defined by $\mathbf{Var}\, X = \mathbb{E}(X^2) - (\mathbb{E}\, X)^2$. Prove that $\mathbf{Var}\, X \geq 0$.

Problem 2. Use Jensen's inequality to prove the AM–GM inequality in full: for positive numbers $a_1, \ldots, a_n$,

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Hint: Take the log of both sides.

Problem 3 (Mantel's Theorem)**.** Let G be a triangle-free graph on n vertices.

- Show that if $uv \in E$, then $\deg u + \deg v \leq n$.
- Show that $\sum_{uv \in E} (\deg u + \deg v) = \sum_{v \in V} \deg^2 v$.
- Use Jensen's inequality to prove that $|E| \leq \frac{n^2}{4}$.

Problem 4 (Kővári–Sós–Turán Theorem)**.** Let G be a bipartite graph with parts A, B where $|A| = |B| = n$. Suppose that G has no copy of $K_{s,t}$.

Let T denote the set of all copies of the star $K_{1,t}$ in G whose center vertex belongs to A.

- Show that $|T| = \sum_{a \in A} \binom{\deg a}{t}$.
- Show that $|T| \leq (s-1)\binom{n}{t}$.
- Use Jensen's inquality to prove that $|E(G)| \leq C(s,t) \cdot n^{2-1/t}$ where $C(s,t)$ is some constant depending only on s, t. Note that the polynomial $\binom{x}{t} = \frac{x(x-1)\cdots(x-t+1)}{t!}$ is convex for $x \geq 0$.