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Solutions must be typeset in L^AT_EX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. This problem is concerned with finding monochromatic trees in t -colorings of $E(K_n)$.

1. Without thinking too hard, show that any t -coloring of $E(K_n)$ has a monochromatic tree on at least $\frac{n}{t}$ many vertices.

We will work to improve this bound.

2. Let G be a bipartite graph with parts A, B .

- (a) Prove that

$$\sum_{\substack{a \in A, b \in B: \\ ab \in E}} (\deg a + \deg b) \geq \left(\frac{1}{|A|} + \frac{1}{|B|} \right) \cdot |E|^2.$$

Hint: The Cauchy–Schwarz inequality states that $(\sum_i a_i b_i)^2 \leq (\sum_i a_i^2)(\sum_i b_i^2)$. In particular, $\sum_{i=1}^n a_i^2 \geq \frac{1}{n} (\sum_{i=1}^n a_i)^2$. Also HW6#2 may be helpful.

- (b) Prove that G contains a tree with at least $(\frac{1}{|A|} + \frac{1}{|B|}) \cdot |E|$ many vertices.

Hint: Consider finding a “double star”, which is two stars with an edge between their center vertices.

3. Show that any t -coloring of $E(K_n)$ contains a monochromatic tree on at least $\frac{n}{t-1}$ many vertices.

Hint: Break into cases depending on whether or not the color- t -graph is connected.

4. For every n which is a multiple of 4, construct a 3-coloring of $E(K_n)$ which *does not* contain a monochromatic tree on $\frac{n}{2} + 1$ many vertices. Thus, the previous part is tight when $t = 3$.

N.b. It is known that the previous part is tight whenever an affine plane of order t exists.