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Solutions must be typeset in $\text{\LaTeX}$. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1. Let T be a tree on at least two vertices. Let $\ell(T)$ denote the number of leaves of T and define $D_{\geq 2} = \{v \in V(T) : \deg v \geq 2\}$. Prove that

$$\ell(T) = 2 + \sum_{v \in D_{\geq 2}} (\deg v - 2).$$

Conclude that $\ell(T) \geq 2$.

Problem 2. Let G be a connected graph and let $w: E(G) \rightarrow \mathbb{R}$ be a weight function. Show that if all weights are distinct, then G has a *unique* minimum spanning tree.

Problem 3. Let G be a bipartite graph with parts A, B . Prove that if

$$\sum_{a \in A} \frac{1}{\deg a} \leq 1,$$

then G has a matching which saturates A . Here, we employ the convention that $\frac{1}{0} = +\infty$.

Problem 4. Let G be a bipartite graph with parts A, B and suppose no vertex of A is isolated. Prove that if all vertices in A have distinct degrees, then G has a matching which saturates A .