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Solutions must be typeset in L^AT_EX. Collaboration is allowed, but must be fully acknowledged. Any collaboration should involve only discussion; all writing **must** be done individually. Use of generative AI is **forbidden**. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (HW1#3 revisited). Use the binomial theorem and generating function concepts to prove that

$$\binom{m_1 + m_2}{n} = \sum_{k=0}^n \binom{m_1}{k} \binom{m_2}{n-k}.$$

Problem 2. Fix $k \geq 2$. Let a_n denote the number of words in $[k]^n$ with an even number of 1's and let b_n denote the number of words in $[k]^n$ with an odd number of 1's.

1. Let $A(z) = \sum_{n \geq 0} a_n z^n$ and $B(z) = \sum_{n \geq 0} b_n z^n$ be their respective generating functions. Show that A and B satisfy the following system of equations:

$$\begin{aligned} A(z) + B(z) &= \frac{1}{1 - kz} \\ (1 - (k-1)z)A(z) - zB(z) &= 1 \end{aligned}$$

2. Show that

$$a_n = \frac{k^n + (k-2)^n}{2}.$$

3. Extra fun (not required): Show that $a_n = \frac{k^n + (k-2)^n}{2}$ without invoking generating functions.

Problem 3 (Derangements, revisited). Let d_n denote the number of derangements of $[n]$.

1. Show that

$$n! = \sum_{i=0}^n \binom{n}{i} d_{n-i}.$$

2. Let $D(z) = \sum_{n \geq 0} \frac{d_n}{n!} z^n$. Use part 1 to show that

$$e^z D(z) = \frac{1}{1-z}.$$

3. Use part 2 to show that

$$d_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$