For a graph G, a *dominating set* is a subset of the vertices $U \subseteq V$, such that for all $v \in V$, either $v \in U$ or v has some neighbor in U. Obviously every graph has a dominating set as we could take U to be all the vertices (in fact, if G is the empty graph, then this is the only dominating set). As such, the largest dominating set is very uninteresting; however, it is very interesting to consider the smallest dominating set. Because we want to avoid cases in which G has isolated vertices, we will consider the minimum degree of G, which is common to denote by $\delta(G)$. Let's first note that if $G = \frac{n}{t} K_t$ (that is G is a collection of disjoint K_t's), then G has no dominating of size smaller than $\frac{n}{t} = \frac{n}{\delta+1}$ as $\delta(G) = t - 1$. As such, we know that in general, we cannot guarantee a dominating set of size smaller than $\frac{n}{\delta+1}$. The next result will show that we can *almost* find a dominating set of that size.

Claim 1. Let G be a graph on n vertices with $\delta(G) = \delta$. Then G has a dominating set of size at most

$$\frac{1 + \log(\delta + 1)}{\delta + 1} n.$$

Proof. Fix $p \in [0, 1]$ to be chose later. Let X be a subset of V where $v \in X$ with probability p. We observe that $\mathbb{E}|X| = np$. We would hope that X is a dominating set, but unfortunately that is a little bit too good to be true. Let Y be the set of all $v \in V$ such that $v \notin X$ *and* v has no neighbor in X. In other words, Y is the set of "bad" vertices which are not dominated by X. Of course, as Y contains all of the "bad" vertices, $X \cup Y$ is a dominating set! As X and Y are disjoint, $\mathbb{E}|X \cup Y| = \mathbb{E}(|X| + |Y|) = \mathbb{E}|X| + \mathbb{E}|Y|$; let's calculate $\mathbb{E}|Y|$. In order for a vertex v to be in Y, v must *not* be in X and also all of v's neighbors must *not* be in X. Hence, as each vertex was placed in X independently,

$$\Pr[v \in Y] = \Pr[v \notin X] \cdot \Pr[N(v) \cap X = \emptyset] = \prod_{u \in N(v) \cup \{v\}} \Pr[u \notin X].$$

The probability that $u \notin X$ is obviously $1 - p$ as u has probability p of being placed in X. Because G had minimum degree δ, $|N(v)| \geq \delta$, so

$$\begin{aligned} \Pr[v \in Y] &= \prod_{u \in N(v) \cup \{v\}} \Pr[u \notin X] \\ &= (1 - p)^{|N(v)|+1} \\ &\leq (1 - p)^{\delta+1} \\ &\leq e^{-p(\delta+1)}. \end{aligned}$$

As such, we find that

$$\mathbb{E}|Y| = \sum_{v \in V} \Pr[v \in Y] \leq \sum_{v \in V} e^{-p(\delta+1)} = ne^{-p(\delta+1)},$$

so $\mathbb{E}|X \cup Y| \leq np + ne^{-p(\delta+1)}$. If $f(p) = np + ne^{-p(\delta+1)}$, then $f'(p) = n - (\delta + 1)ne^{-p(\delta+1)}$, so to optimize our choice of p, we want $(\delta + 1)e^{-p(\delta+1)} = 1$. This holds by picking $p = \frac{\log(\delta+1)}{\delta+1}$. Hence, if we set p to this value,

$$\mathbb{E}|X \cup Y| \leq n\frac{\log(\delta + 1)}{\delta + 1} + ne^{-\frac{\log(\delta+1)}{\delta+1}(\delta+1)} = \frac{1 + \log(\delta + 1)}{\delta + 1} n.$$

As $X \cup Y$ is a dominating set, we have shown that a dominating set of the claimed size must exist. $\square$

We say that a tournament is k-paradoxical if for every k-set of vertices K, there is a $v \notin K$ such that $K \subseteq N^+(v)$. In other words, if we think of $u \to v$ as "u beats v," then we can think of a k-paradoxical tournament in one where if we have k trophies to hand out, then we cannot decide on who to award them to. This is because if we gave the trophies to some k people, then there is someone who beat all of them, so why don't we give them a trophy? It is not obvious that such tournaments exist, but using probabilistic methods, we can show that they do!

Claim 2. For every k, there is an $n = n(k)$, such that there is a k-paradoxical tournament on n vertices.

Proof. Again, we will uniformly at random select a tournament on vertex set $[n]$. For $K \in \binom{[n]}{k}$, let X_K be the random variable that is 1 if there is *no* $v \notin K$ such that $K \subseteq N^+(v)$ and 0 otherwise. Also let $X = \sum_{K \in \binom{[n]}{k}} X_K$, i.e. X is the number of bad k-sets. Of course, as the orientation of each edge was chosen independently,

$$\mathbb{E}X_K = \Pr[\forall v \notin K : K \nsubseteq N^+(v)] = \prod_{v \notin K} \Pr[K \nsubseteq N^+(v)]$$

Now, $\Pr[K \nsubseteq N^+(v)] = 1 - \Pr[K \subseteq N^+(v)]$, and it is easy to observe that $\Pr[K \subseteq N^+(v)] = 2^{-k}$ as it must be the case that for every $u \in K$, $v \to u$. As such,

$$\mathbb{E}X_K = \prod_{v \notin K} (1 - 2^{-k}) = (1 - 2^{-k})^{n-k} \leq e^{-2^{-k}(n-k)}.$$

Hence, by linearity of expectation,

$$\begin{aligned}\mathbb{E}X &= \sum_{K \in \binom{[n]}{k}} \mathbb{E}X_K \\ &= \sum_{K \in \binom{[n]}{k}} e^{-2^{-k}(n-k)} \\ &= \binom{n}{k} e^{-2^{-k}(n-k)} \\ &\leq n^k \cdot e^{-2^{-k}(n-k)}\end{aligned}$$

For a fixed k, we easily observe that $n^k \ll e^{2^{-k}(n-k)}$ as the left hand side is polynomial in n and the right hand side is exponential. As such $n^k \cdot e^{-2^{-k}(n-k)} \to 0$, so there is some $n = n(k)$ for which $n^k \cdot e^{-2^{-k}(n-k)} < 1$, and hence, $\mathbb{E}X < 1$. Note that one can certainly take $n = 4^{k^2}$.

As $\mathbb{E}X < 1$ and X is always a nonnegative integer, there must be some instance in which $X = 0$. As X denotes the number of bad k-sets, we see that there must be a tournament in which every k-set has a vertex that beets it. □

We with a fun little problem that happens to be one of my absolute favorites.

Claim 3. For any 10 points in the plane, there is a way to place non-overlapping (open) unit disks so that each point is contained in some disk.

Proof. Consider the circle packing of the plane shown in Figure 1. This packing is equivalent to tiling the plane with hexagons and inscribing a circle inside of each. With WolframAlpha, it is easy to see that each circle covers at least 91% of the hexagon, so the packing overs at least 91% of the plane. We would like to lay down one of these packings uniformly at random. While it is not necessarily obvious that we can do so, we note that every packing is simply a displacement of the packing centered at $(0, 0)$ by a vector of length at most 1. As we can uniformly select a real number in $[0, 1]$ (the length of the displacement vector) and

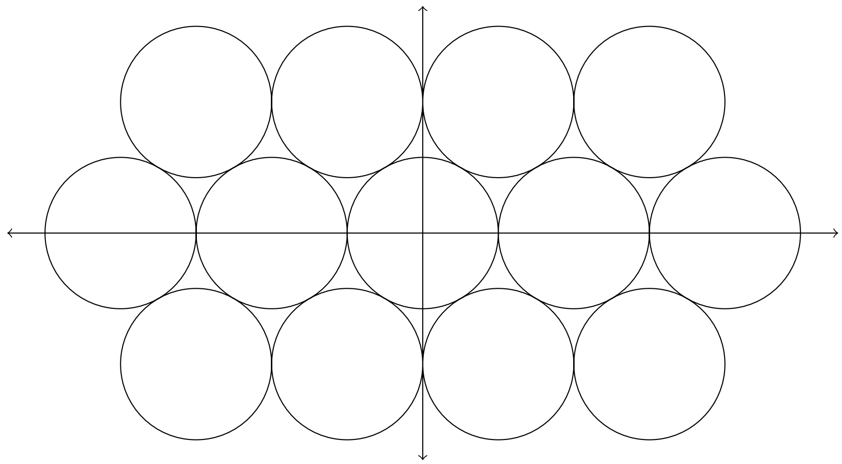

FIGURE 1. The circle packing used in Claim 3.

also uniformly select a real number in $[0, 360)$ (the angle of the displacement vector), we see that we can uniformly sample these circle packings.

Now label the points $1, \ldots, 10$ and let X_i be the random variable which is 1 if point i is covered by the random circle packing and 0 otherwise. As the packing was selected uniformly at random and covers at least 91% of the plane, $\mathbb{E}X_i = \Pr[\text{point } i \text{ is covered}] \geq .91$. Let $X = \sum_{i=1}^{10} X_i$, in other words X is the random variable that counts the number of points that are covered. By linearity of expectation,

$$\mathbb{E}X = \sum_{i=1}^{10} \mathbb{E}X_i \geq \sum_{i=1}^{10} .91 = 9.1.$$

Hence, there must be some instance in which $X \geq 9.1$, so as $X \in \mathbb{Z}^{\geq 0}$, there is an instance in which $X = 10$, i.e. all 10 points are covered. $\square$