As we know from earlier, if $n \geq 6$, then every 2-coloring of the edges of K_n must admit a monochromatic triangle. This yields the following very natural question: for any 2-coloring of the edges of K_n, how many monochromatic triangles must there be?

Claim 1. Asymptotically in n, in any 2-coloring of the edges of K_n, at least 1/4 of the total number of triangles are monochromatic.

Proof. Keep in mind that the total number of triangles in K_n is $\binom{n}{3} \sim \frac{n^3}{6}$.

Let c be any 2-coloring of the edges of K_n and let A denote the number of paths (u, v, w) such that $c(u, v) = 1$ and $c(v, w) = 2$ (i.e. the number red-blue paths of length 3). Additionally, for a vertex v, let $A(v)$ be the number of these red-blue paths such that v is the center vertex. As each red-blue path has a unique center vertex, we observe that $A = \sum_{v \in V} A(v)$. For $i \in [2]$, if we let $\deg_i(v)$ be the number of vertices connected to v by color i, then we observe that $A(v) = \deg_1(v)\deg_2(v)$ as we simply select any red edge and blue edge incident to v.

We now notice that every triangle that is *not* monochromatic must have *exactly* two red-blue paths, so

$$\# \text{ of non-monochromatic triangles} = \frac{1}{2}A.$$

Therefore,

$$\begin{aligned}
\# \text{ of monochromatic triangles} &= \binom{n}{3} - \# \text{ of non-monochromatic triangles} \\
&= \binom{n}{3} - \frac{1}{2}A \\
&= \binom{n}{3} - \frac{1}{2}\sum_{v \in V} \deg_1(v)\deg_2(v).
\end{aligned}$$

Now, as every edge must receive a unique color, we observe that for every v, $\deg_1(v) + \deg_2(v) = n - 1$. Thus, as can be observed through simple calculus, $\sum_{v \in V} \deg_1(v)\deg_2(v)$ is maximized when $|\deg_1(v) - \deg_2(v)| \leq 1$ for all v (in other words, $\deg_1(v) = \lceil (n-1)/2 \rceil$ and $\deg_2(v) = \lfloor (n-1)/2 \rfloor$ or vice versa). Therefore,

$$\begin{aligned}
\# \text{ of monochromatic triangles} &= \binom{n}{3} - \frac{1}{2}\sum_{v \in V} \deg_1(v)\deg_2(v) \\
&\geq \binom{n}{3} - \frac{1}{2}\sum_{v \in V} \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor \\
&= \binom{n}{3} - \frac{1}{2}n \left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor \\
&\sim \frac{n^3}{6} - \frac{n^3}{8} = \frac{n^3}{24}.
\end{aligned}$$

Hence, asymptotically, at least 1/4 of the triangles are monochromatic. □

Corollary 2. *Any 2-coloring of the edges of K_6 must have at least 2 monochromatic triangles.*

Proof. By the previous proof, we know that the number of monochromatic triangles in any 2-coloring of the edges of K_n must have at least $\binom{n}{3} - \frac{n}{2}\left\lceil \frac{n-1}{2} \right\rceil \left\lfloor \frac{n-1}{2} \right\rfloor$ monochromatic triangles. The corollary follows from letting $n = 6$. $\square$

In a directed graph G, a directed path is a set of distinct vertices $(v_1, \ldots, v_k)$ such that $v_i \to v_{i+1}$ for all $i \in [k-1]$.

The *transitive tournament of order* n, which we denote by T_n, is the directed graph on vertex set $[n]$ where $i \to j$ if and only if $i < j$.

Claim 3. Any t-coloring of the edges of T_n must have a monochromatic directed path of length at least $n^{1/t}$.

Proof. Let c be any t-coloring of the edges of T_n. For $i \in [t]$ and a vertex v, define $q_i(v)$ to be the length of the longest i-colored path ending at v. Additionally, let $q(v) = (q_1(v), \ldots, q_t(v))$. Now suppose that $v < u$ are vertices of T_n, then if $c(v, u) = i$, we observe that $q_i(v) + 1 \leq q_i(u)$. In particular, for any vertices $v \neq u$, $q(v) \neq q(u)$ as these vectors differ in at least one component. Now, suppose that L is the length of the longest monochromatic directed path, then we have $q_i(v) \in [L]$ for all i and v; hence, there are precisely L^t choices for $q(v)$. As the $q(v)$'s are all distinct and there are n vertices, we find that $n \leq L^t$, or $L \geq n^{1/t}$ as claimed. $\square$

Problem 4. Show that the above result is tight, i.e. there is a t-coloring of the edges of T_n where no monochromatic directed path has length longer than $n^{1/t}$.

Note that we made strong use of the fact that we were coloring a transitive tournament. If we happened to be coloring a tournament with directed cycles, then the $q(v)$'s need not be distinct, so the argument falls apart. This being said, it turns out that the above claim actually holds for *any* tournament, no matter how many cycles. If we end up covering enough graph theory in this course, in particular talk about chromatic numbers, then hopefully we will have time to prove this more general result as it is one of my absolute favorites.

By applying the above claim, we arrive at the following well-known result.

Theorem 5 (Erdős-Szekeres Theorem). *For any sequence of n distinct real numbers, there is a monotone subsequence of length at least $\sqrt{n}$.*

Proof. Let $a_1, \ldots, a_n$ be a sequence of distinct real numbers; we will use these numbers to give a coloring of the edges of T_n. For $i < j$, let $c(i, j) = 1$ if $a_i < a_j$ and let $c(i, j) = 2$ if $a_i > a_j$. By the previous claim, we know that there is a monochromatic path of length at least $\sqrt{n}$. If this path is in color 1, then the terms associated with the vertices form an increasing subsequence and if the path is in color 2, then the terms associated with the vertices form a decreasing subsequence. In either case, we have a monotone subsequence of length at least $\sqrt{n}$. $\square$