Problem 1. How many words of length n are there coming from $[k]$ (for $k \geq 2$) such that the number 1 is used an even number of times? How many are there where 1 is used an odd number of times?

Let $a(n)$ be the number of words of length n coming from $[k]$ with an even number of 1's and let $b(n)$ be the number of these words with an odd number of 1's. Obviously $a(n) + b(n) = k^n$ and as the empty word contains no 1's, $a(0) = 1$ and $b(0) = 0$.

We observe that $a(n) = b(n-1) + (k-1) \cdot a(n-1)$ as if there are an odd number of 1's in the first $n-1$ positions, then the nth position *must* have a 1, while if there are an even number of 1's in the first $n-1$ positions, then the last position can contain any number between 2 and k. Therefore,

$$a(n) = b(n-1) + (k-1) \cdot a(n-1) = (k-2)a(n-1) + (b(n-1) + a(n-1)) = (k-2)a(n-1) + k^{n-1}.$$

for $n \geq 1$. Let $A(z) = \sum_{n \geq 0} a(n) z^n$ be the generating function for this sequence.

$$\begin{aligned} A(z) &= \sum_{n \geq 0} a(n) z^n = 1 + \sum_{n \geq 1} a(n) z^n \\ &= 1 + \sum_{n \geq 1} \left((k-2)a(n-1) + k^{n-1} \right) z^n \\ &= 1 + z \sum_{n \geq 0} \left((k-2)a(n) + k^n \right) z^n \\ &= 1 + (k-2)z \sum_{n \geq 0} a(n) z^n + z \sum_{n \geq 0} k^n z^n \\ &= 1 + (k-2)zA(z) + \frac{z}{1-kz}. \end{aligned}$$

Hence, $(1-(k-2)z)A(z) = 1 + \frac{z}{1-kz} = \frac{1-(k-1)z}{1-kz}$, so

$$\begin{aligned} A(z) &= \frac{1-(k-1)z}{(1-kz)(1-(k-2)z)} = \frac{2-2kz+2z}{2(1-kz)(1-(k-2)z)} \\ &= \frac{(1-kz) + (1-(k-2)z)}{2(1-kz)(1-(k-2)z)} \\ &= \frac{1}{2(1-kz)} + \frac{1}{2(1-(k-2)z)} \\ &= \frac{1}{2} \sum_{n \geq 0} k^n z^n + \frac{1}{2} \sum_{n \geq 0} (k-2)^n z^n \\ &= \frac{1}{2} \sum_{n \geq 0} (k^n + (k-2)^n) z^n. \end{aligned}$$

We conclude that $a(n) = \frac{1}{2}\left(k^n + (k-2)^n\right)$. Of course, as $a(n) + b(n) = k^n$, we find that $b(n) = \frac{1}{2}\left(k^n - (k-2)^n\right)$. Perhaps surprisingly, if $k \geq 3$, then there are strictly more words with an even number of ones than there are with an odd number of ones (in fact $a(n) - b(n) = (k-2)^n$). Although this is the case, for any fixed $k \geq 2$, we can show that $a(n) \sim b(n)$ (i.e. they are roughly the same for large n).

To see this,

$$\frac{a(n)}{b(n)} = \frac{k^n + (k-2)^n}{k^n - (k-2)^n} = \frac{1 + \left(\frac{k-2}{k}\right)^n}{1 - \left(\frac{k-2}{k}\right)^n},$$

which tends to 1 as $n \to \infty$ for any fixed $k \geq 2$.

Define the partition number of n, denoted $p(n)$, to be the number of ways to write n as the sum of any positive integers. For example, the partitions of 4 are

- 4
- $1+3$
- $2+2$
- $1+1+2$
- $1+1+1+1$

so $p(4) = 5$. Equivalently, $p(n)$ is the number of ways to distribute n identical balls into *any* number of identical bins so that no bin is empty. If $P(z) = \sum_{n\geq 1} p(n)z^n$, then by thinking about how many pieces of a given size are used, we find that

$$P(z) = (1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)(1 + z^3 + z^6 + \cdots)\cdots = \prod_{n\geq 1} \frac{1}{1-z^n}.$$

We can also define $p_d(n)$ to be the number of partitions of n such that every part is distinct. For example 4 has two distinct partitions (4 and $1+3$). We may also define $p_o(n)$ to be the number of partitions of n such that each part is odd. For example 4 has two odd partitions ($1+3$ and $1+1+1+1$). Observe that $p_d(4) = p_o(4)$. This is not a coincidence!

Claim 2. For all $n \in \mathbb{Z}^+$, $p_d(n) = p_o(n)$.

Proof. Let $P_d(z) = \sum_{n\geq 1} p_d(n)z^n$ and $P_o(z) = \sum_{n\geq 1} p_o(n)z^n$ be the respective generating functions of these two sequences. We will show that $P_d(z) = P_o(z)$ from which the claim will follow.

When considering distinct partitions, we can use each size *at most* once, so

$$P_d(z) = (1+z)(1+z^2)(1+z^3)\cdots = \prod_{n\geq 1}(1+z^n).$$

On the other hand, when considering odd partitions, we can only use parts of odd size, so

$$P_o(z) = (1 + z + z^2 + \cdots)(1 + z^3 + z^6 + \cdots)(1 + z^5 + z^{10} + \cdots)\cdots = \prod_{n \text{ odd}} \frac{1}{1-z^n}.$$

By simple algebraic manipulation, we have that

$$\begin{aligned} P_d(z) = \prod_{n\geq 1}(1+z^n) &= \prod_{n\geq 1}\frac{(1+z^n)(1-z^n)}{1-z^n} \\ &= \prod_{n\geq 1}\frac{1-z^{2n}}{1-z^n} \\ &= \frac{\cancel{1-z^2}}{1-z}\frac{\cancel{1-z^4}}{\cancel{1-z^2}}\frac{\cancel{1-z^6}}{1-z^3}\frac{\cancel{1-z^8}}{\cancel{1-z^4}}\frac{\cancel{1-z^{10}}}{1-z^5}\cdots \\ &= \prod_{n \text{ odd}}\frac{1}{1-z^n} = P_o(z) \end{aligned}$$

□

To conclude, here is a fairly easy but fun problem to think about.

Problem 3. Suppose that we were to flip a fair coin n times. Show that, asymptotically in n, the probability that we *never* flipped heads twice in a row is

$$\frac{\varphi^2}{\sqrt{5}}\left(\frac{\varphi}{2}\right)^n \approx 1.17\cdot(0.809)^n$$

where φ is the golden ratio. What if instead we roll a fair k-sided die n times? Show that, asymptotically in n, the probability of never rolling, say, a 1 twice in a row is

$$\frac{1}{2}\left(1+\frac{k+1}{\sqrt{(k-1)(k+3)}}\right)\left(\frac{k-1+\sqrt{(k-1)(k+3)}}{2k}\right)^n.$$