Define the *nth harmonic number* to be $H_n = \sum_{k=1}^{n} \frac{1}{k}$. As we all know from calculus, $H_n \to \infty$ as $n \to \infty$, but we would like to know more about its asymptotic growth.

Claim 1. For all $n \geq 2$, $\log(n+1) < H_n < \log n + 1$.

Proof. To show this, we will approximate $\int_1^n \frac{dx}{x}$. As $1/x$ is monotone decreasing for $x > 0$, we know that the left endpoint Reimann sum upper bounds the integral and that the right endpoint Reimann sum lower bounds it (see Figure 1). As such,

$$\int_1^n \frac{dx}{x} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} = H_{n-1},$$

so $H_n > \int_1^{n+1} \frac{dx}{x} = \log(n+1)$. On the other hand,

$$\int_1^n \frac{dx}{x} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n - 1,$$

so $H_n < \log n + 1$. □

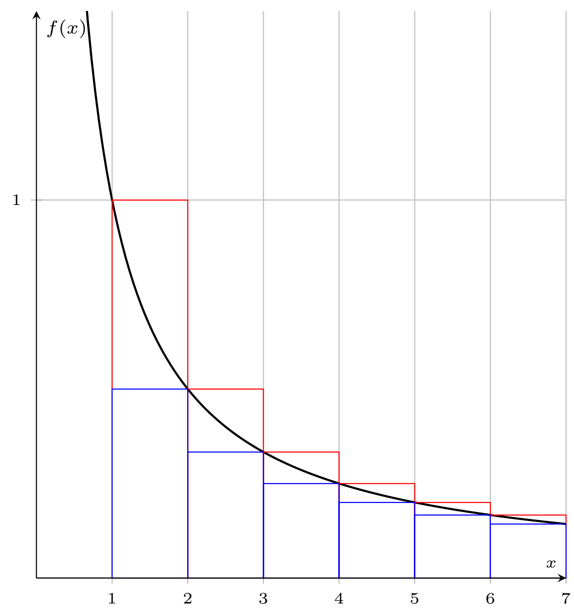

FIGURE 1. The approximations to the integral in Claim 1.

Claim 2. For all $n \geq 1$, $H_n > \log(n+1) + \frac{1}{2}\frac{n}{n+1}$.

Proof. To see this, we will again use the integral estimate of $\int_1^{n+1} \frac{dx}{x}$, but be slightly more careful. For $n \geq 1$, let T_n denote the triangle with vertices $(n, \frac{1}{n}), (n+1, \frac{1}{n}), (n+1, \frac{1}{n+1})$. As $1/x$ is a convex function, we observe that if $n \leq x \leq n+1$, then $1/x$ is bounded above by the line connecting $(n, \frac{1}{n})$ and $(n+1, \frac{1}{n+1})$. As such,

$$\int_1^{n+1} \frac{dx}{x} < H_n - \sum_{i=1}^{n} area(T_i).$$

The claim follows by noting that $area(T_i) = \frac{1}{2}\left(\frac{1}{i} - \frac{1}{i+1}\right)$, so $\sum_{i=1}^{n} area(T_i) = \frac{1}{2}\frac{n}{n+1}$ (which follows from the fact that the sum is telescoping). □

Claim 3. For any $n \in \mathbb{Z}^+$, $n2^{n-1} = \sum_i i\binom{n}{i}$.

Proof. We have already done a double counting proof of this identity, so we will now prove it by using the binomial theorem. The binomial theorem tells us that for all $z \in \mathbb{C}$, $(1+z)^n = \sum_i \binom{n}{i} z^i$. Taking the derivative of both sides (as the right hand side is a finite sum) yields $n(1+z)^{n-1} = \sum_i i\binom{n}{i} z^{i-1}$. The claim follows by setting $z = 1$. □

Claim 4. Let $\mathcal{E}_n = \{S \subseteq [n] : |S| \in 2\mathbb{Z}\}$ and $\mathcal{O}_n = \{S \subseteq [n] : |S| \in 2\mathbb{Z}+1\}$, then $|\mathcal{E}_n| = |\mathcal{O}_n|$ for all $n \geq 1$.

Proof. There is a nice bijection between these sets, namely, if we fix any $T \in \mathcal{O}_n$, then $f_T : S \mapsto S \triangle T$ is a bijection (in fact, involution) from $\mathcal{E}_n$ to $\mathcal{O}_n$.

As an alternate proof, note that $|\mathcal{E}_n| = \sum_{i \text{ even}} \binom{n}{i}$ and $|\mathcal{O}_n| = \sum_{i \text{ odd}} \binom{n}{i}$. By the binomial theorem, we observe that, as $n \geq 1$,

$$0 = (1-1)^n = \sum_i (-1)^i \binom{n}{i} = \sum_{i \text{ even}} \binom{n}{i} - \sum_{i \text{ odd}} \binom{n}{i} = |\mathcal{E}_n| - |\mathcal{O}_n|,$$

so the claim follows. □