For a positive integer n, define $d(n)$ to be the number of divisors of n (including both 1 and n) and let $\bar{d}(n) = \text{ave}_{i\in[n]}\, d(i)$.

Claim 1. For all $n \in \mathbb{Z}^+$, $\log n - \frac{1}{n} < \bar{d}(n) \le \log n + 1$.

Proof. Obviously $\bar{d}(n) = \frac{1}{n}\sum_{i=1}^{n} d(i)$. Consider the $n \times n$ array formed by placing a 1 in position ij if and only if $i \mid j$. Let $\mathcal{O}$ denote the number of 1's in this array, and for $i \in [n]$, let $C(i)$ denote the number of 1's in column i and let $R(i)$ denote the number of 1's in row i. By the definition of the array, $C(i) = d(i)$. On the other hand, we note that the 1's in row i correspond exactly to the multiples of i; hence, $R(i) = \lfloor n/i \rfloor$ as there are precisely that many multiples of i less than or equal to n. We therefore find that

$$\bar{d}(n) = \frac{1}{n}\sum_{i=1}^{n} d(i) = \frac{1}{n}\sum_{i=1}^{n} C(i) = \frac{1}{n}\mathcal{O} = \frac{1}{n}\sum_{i=1}^{n} R(i) = \frac{1}{n}\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor.$$

By noting that $n/i - 1 < \lfloor n/i \rfloor \le n/i$, we observe that

$$\sum_{i=1}^{n} \frac{1}{i} - \frac{1}{n} < \bar{d}(n) \le \sum_{i=1}^{n} \frac{1}{i}.$$

The claim follows from the fact that $\log n \le \sum_{i=1}^{n} 1/i \le \log n + 1$ (which can easily be proved by considering $\int 1/x\; dx$). $\square$

Problem 2. Suppose we have two candidates, Alice and Bob, running for office. It happens to be that Alice receives a votes while Bob receives b votes, where $a > b \ge 1$. Note that any two votes for the same candidate cannot be distinguished. If the votes tallied one at a time in some order, how many ways are there to tally the votes so that Alice and Bob are tied at some nontrivial time?

We begin by noting that any way to tally votes is equivalent to a lattice path from $(0,0)$ to (a,b). To see this, identify a vote for Alice by a step to the right and identify a vote for Bob by a step upward. With this equivalence, it suffices to count the number of lattice paths from $(0,0)$ to (a,b) that touch the main diagonal (i.e. path through a point (i,i) for some integer $i \ge 1$).

Claim 3. Let $a > b \ge 1$. There are precisely $2\binom{a+b-1}{a}$ lattice paths from $(0,0)$ to (a,b) that touch the main diagonal.

Proof. We begin by partitioning the collection of all lattice paths into three parts:

- P_1: The collection of lattice paths that begin with a step to the right and do not touch the main diagonal.
- P_2: The collection of lattice paths that begin with a step to the right and touch the main diagonal.
- P_3: The collection of lattice paths that begin with a step upward.

Notice that every element of P_3 *must* touch (in fact, cross) the main diagonal at some point, as $a > b$. Therefore, the number of lattice paths that touch the main diagonal is precisely $|P_2 \cup P_3| = |P_2| + |P_3|$, so it suffices to determine the sizes of P_2 and P_3.

We will prove that $|P_2| = |P_3| = \binom{a+b-1}{a}$, which will establish the claim. To begin, the lattice paths in P_3 have no restrictions other than beginning with a step upward. Thus, we can easily identify them with the collection of lattice paths from $(0,1)$ to (a,b); hence, $|P_3| = \binom{a+b-1}{a}$.

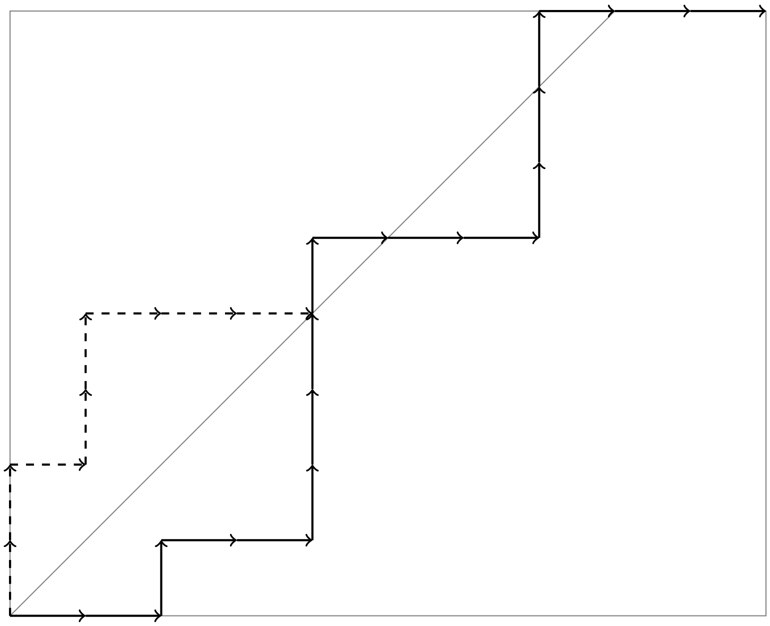

FIGURE 1. An example of the bijection used in Claim 3.

We now establish a bijection between P_2 and P_3, which will conclude the proof. Each element of P_2 *must* touch the main diagonal at some point, so reflect the segment between the beginning and the *first* instance where the path touches the main diagonal (see Figure 1). As the original path began with a step to the right, the new path begins with a step upwards. Further, we have not affected the ending segment of the path, so we still have a path from $(0, 0)$ to (a, b). Hence, we have mapped P_2 into P_3. Beyond this, it is easy to undo this transformation for any element of P_3 by simply reflecting the segment between the beginning at the first instance it touches the main diagonal (as every element of P_3 must touch the main diagonal at some point). Therefore, we have established a bijection between P_2 and P_3 as needed. □