Certainly not every graph is planar, but we might want to ask "how not planar is my graph?" To make this question explicit, we define the following parameter on graphs called the crossing number. Consider a drawing of a graph G in the plane. We say that two edges cross if they overlap in this drawing and we count this as one "crossing." The *crossing number* of a graph G, denoted $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G in the plane (of course, if more than two edges cross at the same point, we get one crossing for each pair). Clearly $\mathrm{cr}(G) = 0$ if and only if G is planar. As a quick exercise, it is pretty easy to find a drawing of K_5 which shows that $\mathrm{cr}(K_5) = 1$ (we know $\mathrm{cr}(K_5) \geq 1$ because K_5 is not planar).

Claim 1. If G is a graph with n vertices and m edges, then $\mathrm{cr}(G) \geq m - 3n$.

Proof. Fix any embedding of G and suppose it has C crossings. This drawing of G can be made planar by removing one edge per crossing; let G' be this graph. As G' was formed by removing at most C edges from G, $|E(G')| \geq m - C$. Also, G' is planar, so from the result in last weeks recitation, $|E(G')| \leq 3n$. Rearranging this inequality, we find that $C \geq m - 3n$. Minimizing over all embeddings of G yields the claim. □

Of course, the bound in Claim 1 is not very good. In particular, it could be the case that "most" edges actually must cross, so conceivably $\mathrm{cr}(G)$ could be close to m^2, where we only proved the lower bound on roughly m. In particular, in the case of K_n, we would (naïvely) expect that $\mathrm{cr}(K_n) = cn^4$ for some constant c as it seems as if all but a constant proportion of the edges must cross. This turns out to be the case, and there is a beautiful proof using probabilistic methods.

Claim 2. If G is a graph with n vertices and m edges, where $m \geq 4n$, then

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Proof. Let us consider selecting a random subgraph of G. Independently, for each $v \in V(G)$, include v in H with probability p (for p to be chosen later). Now fix an embedding of G with $\mathrm{cr}(G)$ many crossings. When considering the random subgraph H, we will simply inherit the embedding from G. Define C_H to be the random variable which counts the number of crossings that H inherits from the specific drawing of G (note that this may be much larger than $\mathrm{cr}(H)$ as even though the embedding may be optimal for G, it probably is not for H). By Claim 1, we know that

$$C_H \geq \mathrm{cr}(H) \geq |E(H)| - 3|V(H)|,$$

(we don't really care about the +6). By linearity of expectation, we have $\mathbb{E}C_H \geq \mathbb{E}|E(H)| - 3\mathbb{E}|V(H)|$. Clearly as each vertex was selected with probability p, $\mathbb{E}|V(H)| = pn$ and $\mathbb{E}|E(H)| = p^2m$ (as H has a given edge if and only if it has both of its endpoints). Furthermore, every crossing is formed by exactly four vertices, so in order for H to inherit a given crossing, it must be the case that each of the four vertices forming this crossing were added to H; hence, $\mathbb{E}C_H = p^4\,\mathrm{cr}(G)$. As such,

$$\mathrm{cr}(G) \geq p^{-2}m - 3p^{-3}n.$$

By basic calculus, we find that the right hand side is optimized when $p = 9n/2m$. In order to have nicer numbers, we will instead just set $p = 4n/m$, which is less than or equal to 1 whenever $m \geq 4n$, which was assumed. The claim follows by plugging in this value for p. □

Notice that we have confirmed our suspicion that $\mathrm{cr}(K_n)$ is quadratic in n. In fact, we showed that $\mathrm{cr}(K_n) \geq 2^{-12}n^4$.