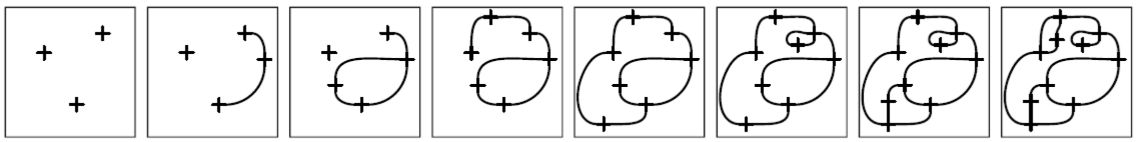
The game of *Brussels Sprouts* was invented by John Conway and is played by starting with n "+" signs drawn in the plane. An "arm" is simply one of the four segments of the + sign. The game proceeds by connecting two open arms (possibly from the same sign) by an edge which crosses no previously drawn edge. Once this new edge has been drawn, we will draw an open arm off each side. An example of several rounds of an unfinished game are shown below with $n = 3$.



The game ends when there is no way to connect two open arms without crossing a previously drawn edge. One natural question to ask is whether or not one can continue playing this game forever.

Claim 1. The game Brussels Sprouts starting with n +'s will always terminate in $5n - 2$ moves.

Proof. Note that we can represent each stage of the game as a planar graph where the +'s are the vertices and the line segments are edges. Of course, this graph may be a multigraph. Let's let G_k be the graph at time k where G_0 is the initial stage with the n vertices and no edges.

Every move in the game creates precisely two new edges and one new vertex, so $|V(G_k)| = n + k$ and $|E(G_k)| = 2k$. Note that at every stage in the game, every face of the graph must have at least one open arm and that there are always precisely $4n$ open arms at any time. Now as a move connects two open arms without crossing a previously draw edge, it must be that these two open arms lie in the same face. Hence, we cannot continue the game if $|F(G_k)| = 4n$ as then every open arm lies in a different face. Hence, the game terminates at time k if $|F(G_k)| = 4n$. As such, if the game terminates at time k, by Euler's formula,

$$2 = |V(G_k)| + |F(G_k)| - |E(G_k)| = n + k + 4n - 2k,$$

or in other words, $k = 5n - 2$. □

Let's talk more about planar graphs. One natural question to ask is "how many edges can a planar graph of order n have?"

Claim 2. If G is a simple planar graph with $n \geq 3$ vertices, then

$$|E| \leq 3n - 6$$

Proof. Let G be a simple planar graph with $n \geq 3$ vertices with the maximum number of edges. Certainly every face of G must be a triangle as otherwise we could connected two vertices on a face and increase the number of edges. Thus, as every edge is contained in precisely two faces and every face contains precisely three edges, $2|E| = 3|F|$. As such, by Euler's formula

$$2 = n + |F| - |E| = n + \frac{2}{3}|E| - |E| = n - \frac{1}{3}|E|,$$

or in other words, $|E| \leq 3n - 6$. □

Claim 2 has an interesting corollary.

Claim 3. If G is a simple planar graph, then $\delta(G) \leq 5$. In fact, G must have at least two vertices of degree at most 5.

Proof. The claim is clearly true for $n \leq 2$, so let G be any simple planar graph of order $n \geq 3$ and consider the average degree $\bar{d}(G)$. By Claim 2,

$$\bar{d}(G) = \frac{1}{n} \sum_{v \in V} \deg(v) = \frac{2|E|}{n} \leq \frac{6n - 12}{n} < 6.$$

As $\bar{d}(G) < 6$, there must be some $v \in V$ with $\deg(v) < 6$, so $\deg(v) \leq 5$. As such $\delta(G) \leq 5$.

Now suppose that G is a planar graph and has only one vertex v with $\deg(v) \leq 5$. Obviously $\deg(v) \geq 1$. Fix an embedding of G in the plane so that v lies on the outside face. Now take multiple copies of this embedding of G, identify their copies of v and arrange the copies so that they do not overlap (which can clearly be done by "sunflowering" them around v). This new graph is planar, but if it consists of enough copies of G around v, then $\deg(v)$ in this new graph can be made arbitrarily large. This is of course a contradiction because this new graph will have minimum degree at least 6, but still be planar. $\square$

This is a very useful fact about planar graphs that will certainly come up later if we have the chance to talk about chromatic numbers. In fact, it will automatically imply that every planar graph is 6-colorable.

We end with a claim that can certainly be proved in many ways, but we choose to prove it in a possibly "strange" way.

Claim 4. If G is a planar graph with $\delta(G) \geq 4$, then G must have a triangle.

Proof. Let's suppose that the claim is false and let G be a counterexample on n vertices (for some n) which has the maximum number of edges. As $\delta(G) \geq 4$, and G is maximum in edges, G is connected and does not have a cut edge (an edge whose deletion disconnects the graph). As such, every edge is incident to exactly two different faces.

For $v \in V$, let $\mu(v) = \deg(v) - 4$ and for $f \in F$, let $\mu(f) = \ell(f) - 4$ where $\ell(f)$ is the number of edges bounding the face. The function μ is often referred to as a *charge function*. By the earlier comment that every edge is incident to exactly two different faces, by simple double counting we observe $\sum_{f \in F} \ell(f) = 2|E|$. Let $\mu(G)$ denote the total amount of charge on G, then by Euler's formula,

$$\mu(G) = \sum_{x \in V \cup F} \mu(x) = \sum_{v \in V} (\deg(v) - 4) + \sum_{f \in F} (\ell(f) - 4) = 2|E| - 4|V| + 2|E| - 4|F| = 4(|E| - |V| - |F|) = -8.$$

However, $\delta(v) \geq 4$ for all $v \in V$, so $\mu(v) \geq 0$. Also, G is a simple graph, so it has no faces of length 1 or 2. Along with the assumption that G is triangle-free, every face has length at least 4, so $\mu(f) \geq 0$ for all $v \in F$. This is a contradiction as the sum of non-negative numbers cannot be negative. $\square$

This claim will immediately imply that all triangle-free planar graphs are 4-colorable. Additionally, the method of applying "charge" and looking for some sort of contradiction is a key part of what is known as a "discharging argument." This method is the only way we know how to prove the general 4-color theorem. I hope to have the chance to walk through some slightly more complicated discharging arguments.