Today we will consider inequalities that can be proved by considering both bounds on the value of a random variable X along with bounds on $\mathbb{E}X$.

Let $[k]^{\leq n}$ be the collection of all words of length at most n coming from $[k]$ and let $[k]^{<\infty}$ be the collection of all finite k-ary words. For a word $w \in [k]^{<\infty}$, we use $\ell(w)$ to denote the length of the word. Let $m < n$, $w \in [k]^m$ and $w' \in [k]^n$. We say that w is a *prefix* of w' if w' agrees with w on the first m digits. A family $\mathcal{F} \subseteq [k]^{<\infty}$ is said to be *prefix-free* if it contains no pair of words where one is a prefix of the other. We would like to get some information about the size of a prefix-free family; however, that is not immediately possible as a prefix-free family can easily have infinite size. Instead, we will prove a density-type argument, which will then give us a bound on its size if we bound the length of the words.

Claim 1. Let $\mathcal{F} \subseteq [k]^{<\infty}$ be a prefix-free family. If $\mathcal{F}_i = \{w \in \mathcal{F} : \ell(w) = i\}$, then

$$\sum_{i \geq 0} \frac{|\mathcal{F}_i|}{k^i} \leq 1.$$

Proof. As $\mathcal{F}$ need not have a uniform bound on the sizes of its elements, we would like to start by uniformly generating an infinite binary string; unfortunately, though, this is impossible. As such, we will show that for every $n \in \mathbb{Z}^+$, $\sum_{i=0}^n |\mathcal{F}_i|/2^i \leq 1$, from which the claim will follow by taking the limit as $n \to \infty$. Fix any $n \in \mathbb{Z}^+$ and uniformly at random select $w \in [k]^n$. For $u \in \mathcal{F}$, let X_u be the random variable which is 1 if u is a prefix of w and 0 otherwise. Also, let $X = \sum_{u \in \mathcal{F}} X_u$, i.e. X is the number of prefixes of w that $\mathcal{F}$ contains (note that X is always finite as only finitely many of the X_u's can be nonzero). We first argue that $X \leq 1$. To see this, certainly if u and u' are both prefixes of w, then either u is a prefix of u' or vice versa; in either case $\mathcal{F}$ can contain at most one, so $X \leq 1$. Of course, as $X \leq 1$, it is also the case that $\mathbb{E}X \leq 1$.

On the other hand for $u \in [k]^{\leq n}$, $\mathbb{E}X_u = \Pr[u \text{ is a prefix of } w] = k^{-\ell(u)}$ and for $u \in [k]^{>n}$, $\mathbb{E}X_u = 0$. Hence, by linearity of expectation,

$$1 \geq \mathbb{E}X = \sum_{u \in \bigcup_{i=0}^n \mathcal{F}_i} \mathbb{E}X_u = \sum_{u \in \bigcup_{i=0}^n \mathcal{F}_i} \frac{1}{k^{\ell(u)}} = \sum_{i=0}^n \frac{|\mathcal{F}_i|}{k^i}.$$

□

Corollary 2. *If $\mathcal{F} \subseteq [k]^{\leq n}$ is prefix-free, then $|\mathcal{F}| \leq k^n$. This is tight.*

Proof. As $\mathcal{F} \subseteq [k]^{\leq n}$, we have that $|\mathcal{F}_i| = 0$ for all $i > n$. Hence, by Claim 1, we have that

$$1 \geq \sum_{i \geq 0} \frac{|\mathcal{F}_i|}{k^i} = \sum_{i=0}^n \frac{|\mathcal{F}_i|}{k^i} \geq \sum_{i=0}^n \frac{|\mathcal{F}_i|}{k^n} = \frac{|\mathcal{F}|}{k^n},$$

from which the claim follows. This bound is achievable by taking $\mathcal{F} = [k]^n$, which is clearly prefix-free and has size k^n. □

For sets A, B, we say that A and B are *comparable* if either $A \subsetneq B$ or $B \subsetneq A$. A family $\mathcal{F}$ of sets is said to be an *antichain* if no pair of sets in $\mathcal{F}$ are comparable. For example, a collection of (finite) sets all having the same size is an antichain. One of the most basic questions to ask is how large an antichain can be if we only take subsets of $[n]$.

Theorem 3 (Sperner's Theorem)**.** *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. This is tight.*

Proof. Fix an antichain $\mathcal{F}$ in $2^{[n]}$. Uniformly select a permutation σ of $[n]$ and consider the chain of subsets

$$\varnothing, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \{\sigma(1), \sigma(2), \sigma(3)\}, \ldots, \{\sigma(1), \ldots, \sigma(n-1)\}, [n].$$

In other words, σ is telling you which element to add in at the ith step. Denote the chain of subsets generated by σ by $\mathcal{C}_\sigma$. For a set $A \in \mathcal{F}$, let X_A be the random variable which is 1 if $A \in \mathcal{C}_\sigma$ and 0 otherwise. Also define $X = \sum_{A \in \mathcal{F}} X_A$, so X is precisely $|\mathcal{F} \cap \mathcal{C}_\sigma|$. To begin, we note that $X \leq 1$ as if $|\mathcal{F} \cap \mathcal{C}_\sigma| \geq 2$, then as every pair of elements of $\mathcal{C}_\sigma$ is comparable, it must be the case that $\mathcal{F}$ would contain some A, B for which $A \subset B$, which is impossible by the fact that $\mathcal{F}$ is an antichain. As $X \leq 1$, we also have the $\mathbb{E}X \leq 1$. Now, for a given $A \in \mathcal{F}$, $\mathbb{E}X_A = \Pr[A \in \mathcal{C}_\sigma] = \frac{|A|!(n-|A|)!}{n!} = 1/\binom{n}{|A|}$. Hence, by linearity of expectation,

$$1 \geq \mathbb{E}X = \sum_{A \in \mathcal{F}} \mathbb{E}X_A = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \geq \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}},$$

from which the claim follows.

The bound is achievable by taking $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$, which is clearly an antichain and has size $\binom{n}{\lfloor n/2 \rfloor}$. □

Let us now look at another interesting family of sets. Let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_k, B_k)\} \subseteq \left(2^{[n]}\right)^2$ be a family with the following properties:

(1) For all $i \in [k]$, $A_i \cap B_i = \varnothing$,
(2) For all $i \neq j \in [k]$, $A_i \cap B_j \neq \varnothing$.

We will refer to such a family as *crossing*.

Claim 4. If $\mathcal{F}$ is a crossing family, then

$$\sum_{(A,B) \in \mathcal{F}} \frac{1}{\binom{|A|+|B|}{|A|}} \leq 1.$$

Proof. For a permutation σ of $[n]$ and subsets A, B, we say that $A \prec_\sigma B$ if all of the elements of A preceed all of the elements of B under the order given by σ. We will say that $\varnothing \prec_\sigma A$ for all nonempty A and permutations σ, though it is not too important. Now choose a permutation σ of $[n]$ uniformly at random and for $(A, B) \in \mathcal{F}$, let $X_{A,B}$ be the random variable which is 1 if $A \prec_\sigma B$ and 0 otherwise. Also, let $X = \sum_{(A,B) \in \mathcal{F}} X_{A,B}$, so X is the number of pairs (A, B) which are nicely ordered under σ. We begin by claiming that $X \leq 1$. To see this, suppose that there were $(A, B) \neq (A', B') \in \mathcal{F}$ such that $A \prec_\sigma B$ and $A' \prec_\sigma B'$. As $\mathcal{F}$ is crossing, there is some $x \in A \cap B'$ and some $y \in A' \cap B$. As $x \in A$ and $y \in B$, we must have $\sigma(x) < \sigma(y)$; however, as $x \in B'$ and $y \in A'$, we must also have $\sigma(y) < \sigma(x)$, which is impossible. Hence, $X \leq 1$, so we also have that $\mathbb{E}X \leq 1$.

On the other hand, as $\mathcal{F}$ is crossing (hence, if $(A, B) \in \mathcal{F}$, then $A \cap B = \varnothing$),

$$\begin{aligned}
\mathbb{E}X_{A,B} &= \Pr[A \prec_\sigma B] \\
&= \frac{\#\text{ of ways to have the elements of } A \text{ preceed those of } B}{n!} \\
&= \frac{\binom{n}{|A|+|B|}|A|!|B|!(n-|A|-|B|)!}{n!} \\
&= \frac{|A|!|B|!}{(|A|+|B|)!} \\
&= \frac{1}{\binom{|A|+|B|}{|A|}}.
\end{aligned}$$

Hence, by linearity of expectation,

$$1 \geq \mathbb{E}X = \sum_{(A,B)\in\mathcal{F}} \mathbb{E}X_{A,B} = \sum_{(A,B)\in\mathcal{F}} \frac{1}{\binom{|A|+|B|}{|A|}}.$$

$\square$