Claim 1. For any 61 points in a disk of radius 4, there is some pair at distance at most $\sqrt{2}$

Proof. Consider laying the unit grid over the disk, as shown in Figure 1. As is obvious from the drawing, none of the corner square touch the disk (of course, we cannot trust a picture, but it is obvious that there is not interaction as $\sqrt{(\pm 3)^2 + (\pm 3)^2} = \sqrt{18} > 4$). Therefore, there are 60 possible squares that our points could lie in. Hence, by the pigeonhole principle, there must be two points in the same square, so the claim follows by noting that a unit square has diameter $\sqrt{2}$. □

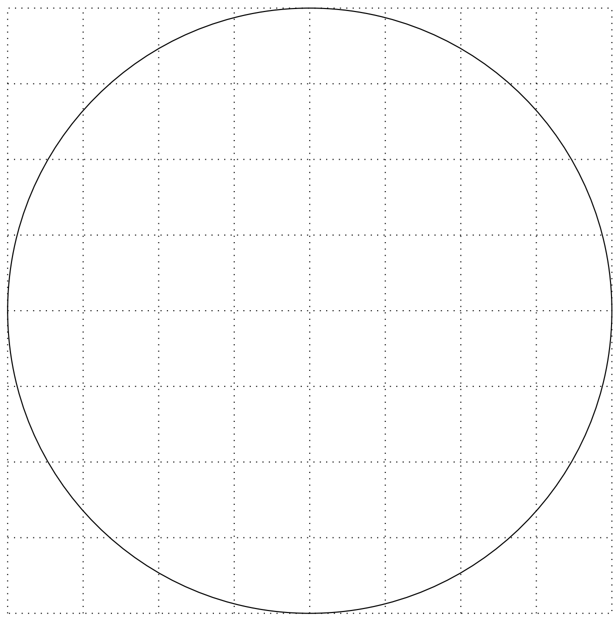

FIGURE 1. The boxes used for Claim 1.

Claim 2. For any 5 points on the surface of a sphere, there is some (closed) hemisphere containing at least 4 of them.

Proof. Cut the sphere into two hemispheres along the line connecting some pair of points. As we care about closed hemispheres, both of these points lie on both hemispheres. As there are three other points, the pigeonhole principle dictates that two must lie on the same hemisphere. Hence, one of these two hemispheres must contain 4 points. □

Problem 1. Suppose we want to form a basket of n fruits consisting of apples and oranges. However, we can only take apples in multiples of 5 and can have at most 4 oranges. How many ways are there to form such a basket?

Of course, by the division algorithm, it is obvious that for any n, there is precisely one way to form such a basket. However, we want to make things overly complicated and give a proof using generating functions. Define $b(n)$ to be the number of ways to form a basket of size n with the above restrictions. Also, let $a(n)$ and $r(n)$ denote the number of ways to make a basket of size n consisting of solely apples or oranges (respectively) also with the above restrictions. Let $B(z) = \sum_{n\geq 0} b(n)z^n$, $A(z) = \sum_{n\geq 0} a(n)z^n$ and

$R(z) = \sum_{n\geq 0} r(n)z^n$ be the ordinary generating functions for each of these sequences. Using what we know about generating functions, it is the case that $B(z) = A(z)R(z)$.

It is easy to observe that $A(z) = \sum_{n\geq 0} z^{5n}$ and $R(z) = 1 + z + z^2 + z^3 + z^4$. Hence,

$$B(z) = (1 + z + z^2 + z^3 + z^4)\sum_{n\geq 0} z^{5n} = \frac{1 - z^5}{1 - z}\frac{1}{1 - z^5} = \frac{1}{1 - z} = \sum_{n\geq 0} z^n.$$

We therefore conclude that $b(n) = [z^n]B(z) = 1$ for all n.