These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec8.pdf

Suppose that G is a triangle-free graph on n vertices; how many edges can G have? Let's first try to build a triangle-free graph with many edges. For integers a, b, the *complete bipartite graph*, denoted $K_{a,b}$ is a bipartite graph with parts A, B where $|A| = a$, $|B| = b$ and every vertex of A is connected to every vertex of B. Observe that $|E(K_{a,b})| = ab$ and that $K_{a,b}$ does not have a triangle. Requiring $a+b = n$ gives us that $|E(K_{a,n-a})| = a(n-a) \leq \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if $a \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$.

It turns out that we can't do any better than this.

Theorem 1 (Mantel's theorem)**.** *If G is a triangle-free graph on n vertices, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if $G = K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.*

There are many, many proofs of Mantel's theorem; furthermore, Mantel's theorem is a special case of the much more general Turán's theorem. For the proof that we'll give today, we need another graph theory notion. For a graph G, a subset $I \subseteq V$ is called an *independent set* if there are no edges between the vertices in I; that is $\binom{I}{2} \cap E = \varnothing$. Note that if G is a bipartite graph with parts A, B, then both A and B are independent sets of G. We denote by $\alpha(G)$ the size of the largest independent set of G.

Lemma 2. *Let G be a triangle-free graph on n vertices with $\alpha(G) = \alpha$. Then $|E(G)| \leq \alpha(n-\alpha)$ with equality if and only if $G = K_{\alpha, n-\alpha}$.*

Since $x(1-x)$ is maximized when $x = 1/2$, Lemma 2 immediately implies Mantel's theorem.

Proof. Let $I \subseteq V$ be an independent set of size α, which we can find since $\alpha(G) = \alpha$. In particular, there are no edges inside of I, so every edge of G has at least one vertex in $V \setminus I$. Therefore,

$$|E| \leq \sum_{v \in V \setminus I} \deg(v). \tag{1}$$

Now, for a vertex $v \in V$, let $N(v)$ denote the set of neighbors of v; that is $N(v) = \{u \in V : uv \in E\}$. In particular, $\deg(v) = |N(v)|$. Since G is triangle-free and v has an edge to every vertex of $N(v)$, it must be the case that $N(v)$ is an independent set. Therefore, $\deg(v) = |N(v)| \leq \alpha$. We conclude that

$$|E| \leq \sum_{v \in V \setminus I} \deg(v) \leq \sum_{v \in V \setminus I} \alpha = \alpha(n-\alpha). \tag{2}$$

We analyze now the case of equality. In (1), we see that equality holds if and only if every edge has *exactly* one vertex in $V \setminus I$. In other words, equality in (1) holds if and only if G is a bipartite graph with parts I and $V \setminus I$. Lastly, the second inequality in (2) holds with equality if and only if $\deg(v) = \alpha$ for all $v \in V \setminus I$. Since $|I| = \alpha$ and we already know that G is bipartite with parts I and $V \setminus I$, this means that every vertex of $V \setminus I$ is connected to every vertex of I. Therefore, $G = K_{|I|, |V \setminus I|} = K_{\alpha, n-\alpha}$. □