These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec6.pdf

Recall Euclid's division theorem: For any $n \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq 1}$, there is a unique $q \in \mathbb{N}$ and $r \in \{0, \dots, a-1\}$ for which $n = kq + r$. Let's prove this using generating functions!

We'll use the following analogy:

We want to form a fruit-basket consisting of $n$ pieces of fruit which can be either apples or oranges. However, we can take apples only in multiples of $k$, and can have at most $k-1$ oranges. How many ways are there to form such a fruit-basket? Denote this number by $b(n)$.

Observe that a proof that $b(n) = 1$ for all $n \in \mathbb{N}$ is equivalent to Euclid's division theorem!

**Claim 1.** *$b(n) = 1$ for all $n \in \mathbb{N}$.*

*Proof.* Let $q(n)$ denote the number of ways to make a fruit-basket of size $n$ consisting only of apples and let $r(n)$ denote the number of ways to make a fruit-basket of size $n$ consisting only of oranges. Let

$$B(z) = \sum_{n\geq 0} b(n)z^n, \qquad Q(z) = \sum_{n\geq 0} q(n)z^n, \qquad R(z) = \sum_{n\geq 0} r(n)z^n,$$

be the generating functions of each of these sequences, so $B(z) = Q(z)R(z)$. Observe that

$$Q(z) = \sum_{n\geq 0} \mathbf{1}[k \mid n]z^n = \sum_{n\geq 0} z^{kn} = \frac{1}{1-z^k},$$

$$R(z) = \sum_{n\geq 0} \mathbf{1}[n \in \{0, 1, \dots, k-1\}]z^n = \sum_{n=0}^{k-1} z^n = \frac{1-z^k}{1-z}.$$

Therefore, $B(z) = Q(z)R(z) = \frac{1}{1-z^k}\frac{1-z^k}{1-z} = \frac{1}{1-z} = \sum_{n\geq 0} z^n$. We conclude that $b(n) = 1$ for all $n \in \mathbb{N}$. □

For a positive integer $n$, the *partition number* of $n$, denoted $p(n)$, is the number of ways to write $n$ as the sum of positive integers (where order doesn't matter). In other words, $p(n)$ is the number of ways to distribute $n$ identical balls into $n$ identical bins (some of which may be empty). For example, $p(4) = 5$ since the partitions of 4 are 4, $1 + 3$, $2 + 2$, $1 + 1 + 2$, and $1 + 1 + 1 + 1$. Additionally, we define $p(0) = 1$ for convenience.

Unlike the number of ways to write $n$ as the sum of positive integers where order matters (which is just stars and bars), determining $p(n)$ is generally a difficult task in the sense that there is no "reasonable" formula. However, we can still write down a generating function.

Define $P(z) = \sum_{n\geq 0} p(n)z^n$. By thinking about how many times of a given number is used in the partition, we find that

$$P(z) = \underbrace{(1 + z + z^2 + \cdots)}_{\text{1's}}\underbrace{(1 + z^2 + z^4 + \cdots)}_{\text{2's}}\underbrace{(1 + z^3 + z^6 + \cdots)}_{\text{3's}}\cdots = \prod_{n\geq 1} \frac{1}{1-z^n}.$$

While this is nice and all, it's not exactly what I want to discuss.

Let $p_d(n)$ denote the number of ways to write $n$ as a sum of *distinct* positive integers (still, order doesn't matter). For example, $p_d(4) = 2$ since the partitions are 4 and $1+3$. Additionally, let $p_o(n)$ denote the number of ways to write $n$ as a sum of *odd* positive integers. For example, $p_d(4) = 2$ since the partitions are $1+3$ and $1+1+1+1$. Again, we define $p_d(0) = p_o(0) = 1$ for convenience.

Observe that $p_d(4) = p_o(4)$; this isn't a coincidence!

**Claim 2.** *For all $n \in \mathbb{N}_{\geq 1}$, we have $p_d(n) = p_o(n)$.*

*Proof.* There is a bijective proof of this fact (sketched below), but let's prove it by using generating functions. Let $P_d(z) = \sum_{n\geq 0} p_d(n)z^n$ and $P_o(z) = \sum_{n\geq 0} p_o(n)z^n$ be the respective generating functions; we need to show that $P_d(z) = P_o(z)$.

When considering distinct partitions, we can use each number at most once, so

$$P_d(z) = \underbrace{(1+z)}_{\text{1's}}\underbrace{(1+z^2)}_{\text{2's}}\underbrace{(1+z^3)}_{\text{3's}}\cdots = \prod_{n\geq 1}(1+z^n).$$

On the other hand, for odd partitions, we can use only odd numbers, so

$$P_o(z) = \underbrace{(1+z+z^2+\cdots)}_{\text{1's}}\underbrace{(1+z^3+z^6+\cdots)}_{\text{3's}}\underbrace{(1+z^5+z^{10}+\cdots)}_{\text{5's}}\cdots = \prod_{n\text{ odd}}\frac{1}{1-z^n}.$$

Now that we have an expression for these generating functions, we can calculate

$$\begin{aligned}P_d(z) &= \prod_{n\geq 1}(1+z^n) = \prod_{n\geq 1}\frac{(1+z^n)(1-z^n)}{(1-z^n)} = \prod_{n\geq 1}\frac{1-z^{2n}}{1-z^n}\\ &= \frac{\cancel{1-z^2}}{1-z}\frac{\cancel{1-z^4}}{\cancel{1-z^2}}\frac{\cancel{1-z^6}}{1-z^3}\frac{\cancel{1-z^8}}{\cancel{1-z^4}}\frac{\cancel{1-z^{10}}}{1-z^5}\cdots\\ &= \prod_{n\text{ odd}}\frac{1}{1-z^n} = P_o(z)\end{aligned}$$ □

For the curious among you, here's a sketch of a bijective proof (with many details missing). I apologize in advance for my cumbersome notation.

*Proof sketch.* Consider a partition of $n$ into odd integers:

$$n = \underbrace{1+\cdots+1}_{\lambda_1}+\underbrace{3+\cdots+3}_{\lambda_3}+\underbrace{5+\cdots+5}_{\lambda_5}+\cdots.$$

Now, consider the binary representation of $\lambda_i$: $\lambda_i = 2^{\lambda_{i1}} + 2^{\lambda_{i2}} + \cdots$ where $\lambda_{i1}, \lambda_{i2}, \ldots$ are distinct. We can expand

$$n = \sum_{i\text{ odd}} i\cdot\lambda_i = \sum_{i\text{ odd}}\sum_j i\cdot 2^{\lambda_{ij}}.$$

Since every positive integer can be written uniquely as a product of a power of 2 and an odd number, we've written $n$ as a sum of distinct positive integers. Concretely, in the case of $n = 4$, this process maps $1+3 \mapsto 1+3$ and $1+1+1+1 \mapsto 4$.

You should check that this process is indeed bijective. □