These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec5.pdf

This is a review day, so hopefully you came prepared with your own questions! Here are a couple additional problems to think about (solutions are on the next page).

Problem 1. Prove that $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}=\binom{m+n}{k}$ in two different ways:

1. Using double counting.

2. Using the binomial theorem.

Problem 2. Fix $k \geq 2$ and let $a_k(n)$ denote the number of words in $[k]^n$ which have an even number of 1's. Find a formula for $a_k(n)$.

(Hint: It may be helpful to consider also $b_k(n)$ to be the number of words in $[k]^n$ which have an odd number of 1's.)

Problem 3. Let Ω be a finite set and let $g\colon 2^\Omega \to \mathbb{R}$ be any function. Define the function $f\colon 2^\Omega \to \mathbb{R}$ by

$$f(S) = \sum_{T \subseteq S} g(T),$$

for all $S \subseteq \Omega$. Prove that for any $S \subseteq \Omega$,

$$g(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f(T).$$

(This is known as the Möbius inversion formula on the Boolean lattice.)

Solution to Problem 1. Part 1 was actually Problem 6 on Homework 1, so please review that solution.

For Part 2, we'll use the binomial theorem. Firstly,

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k.$$

On the other hand,

$$(1+x)^{m+n} = (1+x)^m(1+x)^n = \left(\sum_{k=0}^{m} \binom{m}{k} x^k\right)\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right)$$
$$= \sum_{k=0}^{m+n} \left(\sum_{i=0}^{k} \binom{m}{j}\binom{n}{k-j}\right) x^k,$$

so

$$\sum_{k=0}^{m+1} \binom{m+n}{k} x^k = \sum_{k=0}^{m+n} \left(\sum_{i=0}^{k} \binom{m}{j}\binom{n}{k-j}\right) x^k.$$

Since this holds for every x, it must be the case that for every k, the coefficient of x^k is the same on both sides; therefore, $\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{j}\binom{n}{k-j}$. □

Solution to Problem 2. Let $A_k(n)$ denote the set of words in $[k]^n$ with an even number of 1's and let $B_k(n)$ denote the set of words in $[k]^n$ with an odd number of 1's; therefore $a_k(n) = |A_k(n)|$ and $b_k(n) = |B_k(n)|$. Observe that $A_k(n), B_k(n)$ forms a partition of $[k]^n$ so $a_k(n) + b_k(n) = k^n$. We will show that $a_k(n) - b_k(n) = (k-2)^n$, which will imply that

$$a_k(n) = \frac{1}{2}\big(k^n + (k-2)^n\big).$$

For a word $w \in [k]^n$, let $z(w)$ denote the number of 1's in w. Define the sign of w to be $\sigma(w) = (-1)^{z(w)}$ and set $Q := \sum_{w\in[k]^n} \sigma(w)$. We first observe that

$$Q = \sum_{w\in A_k(n)} (-1)^{\text{even}} + \sum_{w\in B_k(n)} (-1)^{\text{odd}} = a_k(n) - b_k(n).$$

Now, for $w \in [k]^n$, define $F(w) = \min\{i \in [n] : w_i \in \{1,2\}\}$; that is the smallest coordinate of w which is either a 1 or a 2. Observe that $F(w)$ is undefined if and only if w has neither a 1 nor a 2; thus $\{w \in [k]^n : F(w) \text{ undefined}\} = \{3,\dots,k\}^n$, which has size $(k-2)^n$. Furthermore, if $F(w)$ is undefined then $\sigma(w) = 1$ since w has no 1's.

Now, consider w for which $F(w)$ is defined and let $f(w)$ denote the word in which the $F(w)$'th coordinate is flipped from a 1 to a 2 or a 2 to a 1. For instance, $f(342122) = 341122$. Observe that f is an involution on those w's for which $F(w)$ is defined; furthermore the number of 1's in w and $f(w)$ differ by exactly one, so $\sigma(w) = -\sigma(f(w))$.

We can therefore compute

$$Q = \sum_{\substack{w:\\ F(w)\text{ undefined}}} \sigma(w) + \sum_{\{w,f(w)\}} \big(\sigma(w) + \sigma(f(w))\big) = (k-2)^n.$$ □

Solution to Problem 3. We first show that the claimed g is valid.

$$\begin{aligned}
\sum_{T\subseteq S} g(T) &= \sum_{T\subseteq S}\sum_{R\subseteq T}(-1)^{|T|-|R|}f(R) \\
&= \sum_{R\subseteq S}\sum_{T:R\subseteq T\subseteq S}(-1)^{|T|-|R|}f(R) \\
&= \sum_{R\subseteq S} f(R)\Bigg(\sum_{T:R\subseteq T\subseteq S}(-1)^{|T|-|R|}\Bigg) \\
&= \sum_{R\subseteq S} f(R)\cdot \mathbf{1}[R=S] \\
&= f(S).
\end{aligned}$$

Of course, just because the claimed g satisfies the formula, doesn't mean that we're done with the problem. Recall that g was some fixed function in the problem statement and f was defined from g. So far, we've shown only that the claimed formula will define the same f, not that it was actually the function we started with. In other words, we need to show that this is the *only* g that will work.

To this end, suppose that $g, h\colon 2^{\Omega} \to \mathbb{R}$ are such that for every $S \subseteq \Omega$,

$$\sum_{T\subseteq S} g(T) = \sum_{T\subseteq S} h(T);$$

we need to show that $g = h$. Suppose not, then $g(S) \neq h(S)$ for some $S \subseteq \Omega$. Since Ω is a finite set, we may consider the smallest S for which $g(S) \neq h(S)$ (here, smallest simply means smallest size). Note that there may be multiple S's of smallest size; however, we know that $g(T) = h(T)$ for all $|T| < |S|$, which is all that is important. We then compute

$$\begin{aligned}
\sum_{T\subseteq S} g(T) = g(S) + \sum_{T\subsetneq S} g(T) &= g(S) + \sum_{T\subsetneq S} h(T) \\
&\neq h(S) + \sum_{T\subsetneq S} h(T) = \sum_{T\subseteq S} h(T);
\end{aligned}$$

contradicting our original assumption. ☐