These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec2.pdf

In general, we want to prove some identity $LHS = RHS$ (possibly we only know one side and want to simplify the expression). The "double counting" technique works by finding some set Ω such that $|\Omega| = LHS$ and $|\Omega| = RHS$. Of course, finding the set Ω is the hard part!

One technique which could be useful in some situations is to look specifically for an Ω which is a *relation*. Recall that for two sets X, Y, a relation between X and Y is simply a subset of the cartesian product $X \times Y$. In other words, we could try to find some Ω which has the form:

$$\Omega = \big\{(x, y) \in X \times Y : \text{some condition on } x \text{ and } y\big\}.$$

If we can find such an Ω, then there is a natural way to double count: sum over the first coordinate or sum over the second coordinate. Formally,

$$|\Omega| = \sum_{x \in X} |\{y \in Y : (x, y) \in \Omega\}|,$$
$$|\Omega| = \sum_{y \in Y} |\{x \in X : (x, y) \in \Omega\}|.$$

Of course, not every double counting argument can be accomplished by finding a relation, but it's certainly something you can try. Here are some examples.

Claim 1. *For any $n \geq r \in \mathbb{N}$, we have*

$$\sum_{k=r}^{n} \binom{n}{k}\binom{k}{r} = 2^{n-r}\binom{n}{r}.$$

Proof. Intuitively. $\binom{n}{k}\binom{k}{r}$ counts the number of ways to pick a k-set from an n-set and then pick an r-set from this k-set. In other words,

$$\binom{n}{k}\binom{k}{r} = \left|\left\{(A, B) \in \binom{[n]}{k} \times \binom{[n]}{r} : B \subseteq A\right\}\right|.$$

Since the LHS is a sum over k, a natural candidate for Ω is

$$\Omega = \left\{(A, B) \in 2^{[n]} \times \binom{[n]}{r} : B \subseteq A\right\},$$

which is a relation. By summing over the first coordinate, we have

$$|\Omega| = \sum_{A \in 2^{[n]}} \left|\left\{B \in \binom{[n]}{r} : B \subseteq A\right\}\right| = \sum_{A \in 2^{[n]}} \binom{|A|}{r}$$
$$= \sum_{k=0}^{n} \sum_{A \in \binom{[n]}{k}} \binom{k}{r} = \sum_{k=0}^{n} \binom{n}{k}\binom{k}{r} = \sum_{k=r}^{n} \binom{n}{k}\binom{k}{r}.$$

By summing over the second coordinate, we have

$$
\begin{aligned}
|\Omega| &= \sum_{B\in\binom{[n]}{r}} \left|\left\{A \in 2^{[n]} : B \subseteq A\right\}\right| = \sum_{B\in\binom{[n]}{r}} \left|2^{[n]\setminus B}\right| \\
&= \sum_{B\in\binom{[n]}{r}} 2^{n-r} = 2^{n-r}\binom{n}{r}. \qquad \square
\end{aligned}
$$

Claim 2. *For $n \in \mathbb{N}_{\geq 1}$, let $d(n)$ denote the number of divisors of n (including 1 and n). Then for any $n \in \mathbb{N}_{\geq 1}$,*

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor.$$

Proof. Observe that $d(k) = |\{i \in \mathbb{N}_{\geq 1} : i \mid k\}|$. Since we're summing over all $k \in [n]$, a natural candidate for Ω is

$$\Omega = \left\{(i,k) \in \mathbb{N}_{\geq 1} \times [n] : i \mid k\right\},$$

which is a relation. By partitioning based on the second coordinate, we have

$$|\Omega| = \sum_{k\in[n]} |\{i \in \mathbb{N}_{\geq 1} : i \mid k\}| = \sum_{k\in[n]} d(k) = \sum_{k=1}^{n} d(k).$$

By partitioning based on the first coordinate, we have

$$
\begin{aligned}
|\Omega| &= \sum_{i\in\mathbb{N}_{\geq 1}} |\{k \in [n] : i \mid k\}| = \sum_{i\in\mathbb{N}_{\geq 1}} |\{\ell \in \mathbb{N}_{\geq 1} : i \cdot \ell \in [n]\}| \\
&= \sum_{i\in\mathbb{N}_{\geq 1}} \max\{\ell \in \mathbb{N} : i \cdot \ell \leq n\} = \sum_{i\in\mathbb{N}_{\geq 1}} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor. \qquad \square
\end{aligned}
$$

Using some facts that we'll prove later in the class, this implies that the average number of divisors of the integers in $[n]$ is approximately $\log n$.

For the next problem, recall that a permutation of $[n]$ is simply a bijection $\pi\colon [n] \to [n]$. The set of all permutations of $[n]$ is denoted by S_n (S is used since this is called the *symmetric group* on n elements). A *fixed point* of π is an element $x \in [n]$ for which $\pi(x) = x$.

Claim 3. *For $n \in \mathbb{N}_{\geq 1}$ and $k \in [n]$, let $p_n(k)$ denote the number of permutations of $[n]$ which have exactly k fixed points. Then*

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

Proof. Of course, a natural candidate to double count is S_n since clearly $|S_n| = n!$. For the LHS, it would then be natural to partition S_n based on the number of fixed points. Unfortunately, this would lead only to the identity $n! = \sum_{k=0}^{n} p_n(k)$, which is not what we're looking for...

Instead, let's try to find a natural candidate for Ω by looking at the LHS. For ease of notation, let $S_n(k)$ denote the set of permutations of $[n]$ which have exactly k fixed points, so that $|S_n(k)| = p_n(k)$. Intuitively,

$$k \cdot p_n(k) = \left|\left\{(\pi, x) \in S_n(k) \times [n] : \pi(x) = x\right\}\right|.$$

Thus, a natural candidate for Ω is

$$\Omega = \big\{(\pi, x) \in S_n \times [n] : \pi(x) = x\big\},$$

which is a relation. By partitioning based on the first coordinate, we have

$$\begin{aligned} |\Omega| &= \sum_{\pi \in S_n} |\{x \in [n] : \pi(x) = x\}| = \sum_{\pi \in S_n} (\#\text{ fixed points of } \pi) \\ &= \sum_{k=0}^{n} \sum_{\pi \in S_n(k)} k = \sum_{k=0}^{n} k|S_n(k)| = \sum_{k=0}^{n} k \cdot p_n(k). \end{aligned}$$

By partitioning based on the second coordinate, we have

$$\begin{aligned} |\Omega| &= \sum_{x \in [n]} |\{\pi \in S_n : \pi(x) = x\}| = \sum_{x \in [n]} (\#\text{ permutations of } [n] \setminus \{x\}) \\ &= \sum_{x \in [n]} (n-1)! = n \cdot (n-1)! = n!. \end{aligned}$$

□

Later in the class, we'll prove the nice fact that $p_n(0) = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor$ where e is the base of the natural logarithm. Permutations with no fixed points are called *derangements*.