These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec1.pdf

For non-negative integers n, k, define the set

$$A(n,k) := \{(x_1, \dots, x_k) \in \mathbb{N}^k : x_1 + \cdots + x_k = n\}.$$

Claim 1. *The number of ways to color n indistinguishable balls using k distinct colors (not every color must be used) is precisely $|A(n,k)|$.*

Proof. Denote by $C(n,k)$ the set of all k-colorings of n indistinguishable balls. We show that $|C(n,k)| = |A(n,k)|$ by finding a bijection $f\colon C(n,k) \to A(n,k)$.

Fix $c \in C(n,k)$ and for $i \in [k]$, let c_i denote the number of balls of color i in c. Observe that $c_i \in \mathbb{N}$ for all i and that $c_1 + \cdots + c_k = n$ since there are n balls in total, each of which has a color. Thus, define $f(c) = (c_1, \dots, c_n)$. By the observation, f is indeed a map from $C(n,k)$ to $A(n,k)$; we must show it is both injective and surjective.

Surjective: Consider any $(x_1, \dots, x_k) \in A(n,k)$; we must find some coloring $c \in C(n,k)$ for which $f(c) = (x_1, \dots, x_k)$. Consider an arbitrary ordering of the n balls. Define c by coloring the first x_1 balls with color 1, the next x_2 balls with color 2, etc. Since $x_i \in \mathbb{N}$ and $x_1 + \cdots + x_k = n$, this is a valid coloring, and $f(c) = (x_1, \dots, x_n)$.

Injective: Consider any $c \neq c' \in C(n,k)$; we must show that $f(c) \neq f(c')$. Since $c \neq c'$ and the balls are indistinguishable, there must be some color $i \in [k]$ for which $c_i \neq c_i'$; hence $(c_1, \dots, c_k) \neq (c_1', \dots, c_k')$ and so $f(c) \neq f(c')$. □

So, how big is $A(n,k)$?

Claim 2. $|A(n,k)| = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

Proof. We will prove this through what is known as the "stars and bars" argument. Let $B(n,k)$ denote the set of all ways to arrange n indistinguishable stars ($*$) and $k-1$ indistinguishable bars ($|$) in a row. For instance, with $n = k = 5$, $||**|*|**$ is an element of $B(n,k)$. Observe that we can think of $B(n,k)$ in the following way: there are $n+k-1$ empty slots, each of which can be filled with either a $*$ or a $|$. Since an element of $B(n,k)$ is determined uniquely by the positions of the bars (since then every other slot must then contain a $*$), we have $|B(n,k)| = \binom{n+k-1}{k-1}$ since there are precisely this many ways to place $k-1$ bars into $n+k-1$ slots. Equivalently, an element of $B(n,k)$ is determined uniquely by the positions of the stars, so $|B(n,k)| = \binom{n+k-1}{n}$ by the same reasoning.

Thus, it suffices to establish a bijection $f\colon A(n,k) \to B(n,k)$. For $(x_1, \dots, x_k) \in A(n,k)$, we form an arrangement of stars and bars by first filling the first x_1 slots with stars and place a bar, then fill the next x_2 slots with stars and place a bar, etc. This continues until we finally place x_k stars, but we do not place a bar after them. Let $f(x_1, \dots, x_k)$ denote this arrangement; for example, $f(0,3,0,0,1) = |***|||*$.

Since $x_1 + \cdots + x_k = n$, we have placed n stars and $k-1$ bars, we have $f(x_1, \dots, x_k) \in B(n,k)$. We must show now that f is a bijection.

Surjective: Consider an arrangement b of stars and bars in $B(n,k)$; we must find some $(x_1,\dots,x_k) \in A(n,k)$ for which $f(x_1,\dots,x_k) = b$. Let x_1 denote the number of stars which appear before the first bar (which may be 0), let x_k denote the number of stars which appear after the last bar, and for all other i, let x_i denote the number of stars which appear between the $(i-1)$st bar and the ith bar. Since there are $k-1$ bars, each of these numbers are well-defined and are non-negative integers. Furthermore, since there are n stars in total, $x_1 + \cdots + x_k = n$, so $(x_1,\dots,x_k) \in A(n,k)$. Finally, by construction, $f(x_1,\dots,x_k) = b$.

Injective: Suppose that $(x_1,\dots,x_k), (x'_1,\dots,x'_k) \in A(n,k)$ have $f(x_1,\dots,x_k) = f(x'_1,\dots,x'_k)$; we must show that $x_i = x'_i$ for all $i \in [k]$. By construction, the arrangement $f(x_1,\dots,x_k)$ has x_1 stars before the first bar, x_k stars after the last bar, and x_i stars between the $(i-1)$st and ith bars for all other i. Similarly, $f(x'_1,\dots,x'_k)$ has x'_1 stars before the first bar, x'_k stars after the last bar, and x'_i stars between the $(i-1)$st and ith bars for all other i. We conclude that $x_i = x'_i$ for all $i \in [k]$. □

For non-negative integers n,k, define the set

$$A'(n,k) := \big\{(x_1,\dots,x_k) \in \mathbb{N}^k : x_1 + \cdots + x_k = n,\ x_i \geq 1\big\}.$$

Following the same ideas as in Claim 1, we see that $|A'(n,k)|$ is precisely the number of ways to color n indistinguishable balls using k distinct colors so that each color is used at least once. So, how big is $A'(n,k)$?

Claim 3. $|A'(n,k)| = |A(n-k,k)| = \binom{n-1}{k-1}$.

Proof. We already know that $|A(n-k,k)| = \binom{n-1}{k-1}$ through Claim 1, so it is enough to show that $|A'(n,k)| = |A(n-k,k)|$, which we will do by finding a bijection $f\colon A'(n,k) \to A(n-k,k)$. For $(x_1,\dots,x_k) \in A'(n,k)$, define

$$f(x_1,\dots,x_k) = (x_1-1,\dots,x_k-1).$$

Since each x_i is a positive integer, $x_i - 1$ is a non-negative integer and $(x_1-1)+\cdots+(x_k-1) = (x_1+\cdots+x_k)-k = n-k$; hence f is indeed a map from $A'(n,k)$ to $A(n-k,k)$. We must show that f is bijective.

Surjective: Consider $(y_1,\dots,y_k) \in A(n-k,k)$; we must find $(x_1,\dots,x_k) \in A'(n,k)$ with $f(x_1,\dots,x_k) = (y_1,\dots,y_k)$. Consider $x_i = y_i+1$; certainly $x_i \in \mathbb{N}_{\geq 1}$ since $y_i \in \mathbb{N}$ and $x_1+\cdots+x_k = (y_1+1)+\cdots+(y_k+1) = n-k+k = n$, so $(x_1,\dots,x_k) \in A'(n,k)$. Furthermore, by definition, $f(x_1,\dots,x_k) = (y_1,\dots,y_k)$.

Injective: Suppose that $(x_1,\dots,x_k), (x'_1,\dots,x'_k) \in A'(n,k)$ have $f(x_1,\dots,x_k) = f(x'_1,\dots,x'_k)$. Thus, $(x_1-1,\dots,x_k-1) = (x'_1-1,\dots,x'_k-1)$ and so $(x_1,\dots,x_k) = (x'_1,\dots,x'_k)$ as needed. □

A similar argument shows that for $\ell_1,\dots,\ell_k \in \mathbb{N}$,

$$\big|\big\{(x_1,\dots,x_k) \in \mathbb{N}^k : x_1+\cdots+x_k = n,\ x_i \geq \ell_i\big\}\big| = |A(n-\ell_1-\cdots-\ell_k,k)| = \binom{n+k-\ell_1-\cdots-\ell_k-1}{k-1}.$$

In Claim 3, we showed that $|A'(n,k)| = \binom{n-1}{k-1}$. The number $\binom{n-1}{k-1}$ is additionally the number of subsets of $[n-1]$ of size $k-1$... Can we find a direct bijection between $A'(n,k)$ and these subsets?

Firstly, some notation. For a set X and a non-negative integer k, we denote the set of all k-subsets of X by

$$\binom{X}{k} := \big\{K \subseteq X : |K| = k\big\}.$$

We use this notation since $\left|\binom{X}{k}\right| = \binom{|X|}{k}$ for any finite set X.

Claim 4. *There is a direct bijection from $A'(n,k)$ to $\binom{[n-1]}{k-1}$.*

Proof. For $(x_1, \dots, x_k) \in A'(n,k)$, define

$$f(x_1,\dots,x_k) = \left\{\sum_{j=1}^{i} x_i : i \in [k-1]\right\} = \big\{x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3,\ \dots,\ x_1 + \cdots + x_{k-1}\big\}.$$

Notice that we only consider up to the $(k-1)$st partial sum and not the sum of all x_i. Intuitively, this is because we know that $x_1 + \cdots + x_k = n$, so this would be redundant.

We must first argue that $f(x_1,\dots,x_k) \in \binom{[n-1]}{k-1}$. To begin, each x_i is a positive integer, and so each $\sum_{j=1}^{i} x_j$ is also a positive integer. Furthermore, for any $i \in [k-1]$, we have

$$1 \le x_1 \le \sum_{j=1}^{i} x_j \le \sum_{j=1}^{k-1} x_j = n - x_k \le n-1.$$

Thus, $f(x_1,\dots,x_k)$ is indeed a subset of $[n-1]$; we still must show that it has size $k-1$. By definition, $f(x_1,\dots,x_k)$ has at most $k-1$ elements, but we could run into trouble if there were some $i \neq \ell \in [k-1]$ for which $\sum_{j=1}^{i} x_j = \sum_{j=1}^{\ell} x_j$ since then we would have written the same number twice. To show that this does not happen, we observe that for any $i \in [k-2]$,

$$\sum_{j=1}^{i} x_j < \sum_{j=1}^{i} x_j + 1 \le \sum_{j=1}^{i+1} x_j.$$

Therefore, f is indeed a map from $A'(n,k)$ to $\binom{[n-1]}{k-1}$. We must now argue that f is bijective.

Surjective: Consider any $S \in \binom{[n-1]}{k-1}$; we must find some $(x_1,\dots,x_k) \in A'(n,k)$ for which $f(x_1,\dots,x_k) = S$. Suppose that $S = \{s_1,\dots,s_{k-1}\}$ where $s_1 < \cdots < s_{k-1}$. For convenience, additionally set $s_k = n$. Define $x_1 = s_1$, $x_2 = s_2 - s_1$, $x_3 = s_3 - s_2$, etc. In other words, for $i \in \{2,\dots,k\}$, we define $x_i = s_i - s_{i-1}$ and also $x_1 = s_1$. We claim that $(x_1,\dots,x_k) \in A'(n,k)$ and that $f(x_1,\dots,x_k) = S$.

To do this, we prove first by induction on $i \in [k]$ that $\sum_{j=1}^{i} x_j = s_i$. We defined $x_1 = s_1$, so the base case is clear. Suppose now that the claim holds for some $i \in [k-1]$; we need to prove that $\sum_{j=1}^{i+1} x_j = s_{i+1}$. To see this,

$$\sum_{j=1}^{i+1} x_j = \sum_{j=1}^{i} x_j + x_{i+1} = s_i + (s_{i+1} - s_i) = s_{i+1}.$$

Therefore, if it is the case that $(x_1, \ldots, x_k) \in A'(n,k)$, we have shown that $f(x_1, \ldots, x_k) = S$. To show that $(x_1, \ldots, x_k) \in A'(n,k)$, we need to show that each x_i is a positive integer and that $x_1 + \cdots + x_k = n$. The latter is true since we have already shown that $x_1 + \cdots + x_k = s_k = n$. For the former, $x_1 = s_1 \in [n-1]$ and so $x_1 \in \mathbb{N}_{\geq 1}$. Furthermore, $s_i \in [n]$ for all $i \in [k]$ and also $s_i > s_{i-1}$, so $x_i = s_i - s_{i-1} \geq 1$ and $x_i \in \mathbb{N}$.

Injective: Suppose that $(x_1, \ldots, x_k) \neq (x'_1, \ldots, x'_k) \in A'(n,k)$; we must show that $f(x_1, \ldots, x_k) \neq f(x'_1, \ldots, x'_k)$. Since $(x_1, \ldots, x_k) \neq (x'_1, \ldots, x'_k)$, there is some $i \in [k]$ for which $x_i \neq x'_i$; let i denote the *smallest* such index for which this holds (so that $x_j = x'_j$ for all $j < i$). Therefore, $\sum_{j=1}^{\ell} x_j = \sum_{j=1}^{\ell} x'_j$ for all $\ell < i$; this additionally implies that $\sum_{j=1}^{i} x_j \neq \sum_{j=1}^{i} x'_j$. Observe that $i \in [k-1]$. Indeed, if $i = k$, then $x_1 + \cdots + x_{k-1} = x'_1 + \cdots + x'_{k-1}$, but we know also that $x_1 + \cdots + x_k = n = x'_1 + \cdots + x'_k$, which implies that $x_k = x'_k$ as well; a contradiction.

Without loss of generality, suppose that $\sum_{j=1}^{i} x_j < \sum_{j=1}^{i} x'_j$.

Set $s_\ell = \sum_{j=1}^{\ell} x_j$ and $s'_\ell = \sum_{j=1}^{\ell} x'_j$ for all $\ell \in [k-1]$; thus $f(x_1, \ldots, x_k) = \{s_1, \ldots, s_{k-1}\} =: S$ and $f(x'_1, \ldots, x'_k) = \{s'_1, \ldots, s'_k\} =: S'$. By our previous observations, we know that $s_1 < \cdots < s_{k-1}$ and $s'_1 < \cdots < s'_{k-1}$; however, $s_j = s'_j$ for all $j < i$ by the definition of i. We thus have $s'_1 < \cdots < s'_{i-1} < s_i < s'_i < s'_{i+1} < \cdots < s'_{k-1}$; therefore, $s_i \in S$, but $s_i \notin S'$, implying that $S \neq S'$ as needed. $\square$