This quiz is from http://math.cmu.edu/~cocox/teaching/discrete20/quiz12.pdf

Problem 1. Prove that G is a connected, 2-regular graph if and only if G is a cycle.

Problem 2. There are $n \geq 3$ participants in an event. Each of these participants know at least $n/2$ other participants. Show that there is a way to seat the participants around a circular table so that each participant knows both people seated next to them.

Problem 3. Let G be any graph. A *cycle decomposition* of G is a collection of cycles $C_1, \ldots, C_k$ that partition the edge-set of G; that is $E = \bigsqcup_{i=1}^{k} E(C_i)$. Note that in a cycle decomposition, the cycles can share vertices, but they cannot share edges.

Show that G has a cycle decomposition if and only if every vertex of G has even degree.

Problem 4. Let G be a graph. Let $\operatorname{conn}(G)$ denote the set of connected components of G (e.g. $\operatorname{conn}(G) = \{G\}$ if and only if G is connected). For a subset $U \subseteq V$, let $G - U$ denote the graph formed by deleting the vertices in U from G: formally, $V(G-U) = V \setminus U$ and $E(G-U) = E \cap \binom{V \setminus U}{2}$.

Show that if G is Hamiltonian, then $|\operatorname{conn}(G-U)| \leq |U|$ for all non-empty $U \subseteq V$.

Problem 5 (Bonus)**.** Let G be a graph. For a subset $A \subseteq V$ and a vertex $v \in V$, define $\deg_A(v) = \big|\{u \in A : \{u, v\} \in E\}\big|$. Consider the following algorithm whose input is a graph $G = (V, E)$:

procedure BiPartition(G)
  $V_0 \leftarrow V$
  $V_1 \leftarrow \varnothing$
  while there exists $v \in V_i$ such that $\deg_{V_{1-i}}(v) < \deg(v)/2$ **do**
    $V_i \leftarrow V_i \setminus \{v\}$
    $V_{1-i} \leftarrow V_{1-i} \cup \{v\}$
  end while
  return (V_0, V_1)
end procedure

Prove the following:

1. BiPartition(G) eventually terminates and returns a pair (V_0, V_1) where $V = V_0 \sqcup V_1$.

   (Hint: Show the algorithm terminates after at most $|E|$ iterations of the while loop)

2. If BiPartition(G) $= (V_0, V_1)$, then G has at least $|E|/2$ edges between V_0 and V_1.

(Note: This yields a polynomial-time algorithm to find the subgraph in HW7(5))