These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/pie.pdf

Let Ω be a finite set. For a subset $X \subseteq \Omega$, the indicator function for X is the function $\mathbf{1}_X \colon \Omega \to \mathbb{R}$ where $\mathbf{1}_X(x) = 1$ if $x \in X$ and $\mathbf{1}_X(x) = 0$ if $x \notin X$. Observe that

$$|X| = \sum_{x \in \Omega} \mathbf{1}_X(x).$$

Let $B_1, \ldots, B_n \subseteq \Omega$ and for $S \subseteq [n]$, define $B_S \overset{\text{def}}{=} \bigcap_{i \in S} B_i$. For $m \in \{0, \ldots, n\}$ define the function $f_m \colon \Omega \to \mathbb{R}$ by

$$f_m(x) \overset{\text{def}}{=} \sum_{S \in \binom{[n]}{\leq m}} (-1)^{|S|} \mathbf{1}_{B_S}(x),$$

where $\binom{[n]}{\leq m} \overset{\text{def}}{=} \{S \subseteq [n] : |S| \leq m\}$. Additionally, set $f \overset{\text{def}}{=} f_n$, so that

$$f(x) = \sum_{S \subseteq [n]} (-1)^{|S|} \mathbf{1}_{B_S}(x).$$

Now, for $x \in \Omega$, define

$$N_x \overset{\text{def}}{=} \{i \in [n] : x \in B_i\},$$

and observe that $x \in B_S$ if and only if $S \subseteq N_x$.

The inclusion-exclusion formula will follow almost immediately from the following lemma.

Lemma 1. *$f(x) = \mathbf{1}_{\Omega \setminus \bigcup_{i \in [n]} B_i}(x)$ for all $x \in \Omega$.*

Proof. Fix $x \in \Omega$. We compute

$$f(x) = \sum_{S \subseteq [n]} (-1)^{|S|} \mathbf{1}_{B_S}(x) = \sum_{S \subseteq N_x} (-1)^{|S|} = \sum_{k=0}^{|N_x|} \binom{|N_x|}{k} (-1)^k.$$

The binomial theorem tells us that $f(x) = 1$ if $|N_x| = 0$ and $f(x) = 0$ otherwise. Since $|N_x| = 0$ if and only if $x \notin \bigcup_{i \in [n]} B_i$, the claim follows. □

Theorem 2 (Inclusion-exclusion). $\left| \Omega \setminus \bigcup_{i \in [n]} B_i \right| = \sum_{S \subseteq [n]} (-1)^{|S|} |B_S|.$

Proof. Using the lemma above, we compute,

$$\begin{aligned} \left| \Omega \setminus \bigcup_{i \in [n]} B_i \right| &= \sum_{x \in \Omega} \mathbf{1}_{\Omega \setminus \bigcup_{i \in [n]} B_i}(x) = \sum_{x \in \Omega} f(x) = \sum_{x \in \Omega} \sum_{S \subseteq [n]} (-1)^{|S|} \mathbf{1}_{B_S}(x) \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} \sum_{x \in \Omega} \mathbf{1}_{B_S}(x) = \sum_{S \subseteq [n]} (-1)^{|S|} |B_S|. \end{aligned}$$

□

Similarly, the proof of the Bonferonni inequalities will follow almost immediately from the following lemma.

Lemma 3. *For $x \in \Omega$ and $m \in \{0, \ldots, n\}$,*

$$f_m(x) \geq \mathbf{1}_{\Omega \setminus \bigcup_{i \in [n]} B_i}(x) \qquad \textit{for } m \textit{ even,}$$
$$f_m(x) \leq \mathbf{1}_{\Omega \setminus \bigcup_{i \in [n]} B_i}(x) \qquad \textit{for } m \textit{ odd.}$$

Proof. Fix $x \in \Omega$. We compute

$$f(x) = \sum_{S \in \binom{[n]}{\leq m}} (-1)^{|S|} \mathbf{1}_{B_S}(x) = \sum_{S \in \binom{N_x}{\leq m}} (-1)^{|S|} = \sum_{k=0}^{m} \binom{|N_x|}{k} (-1)^k.$$

If $|N_x| = 0$, then certainly $f(x) = 1$. On the other hand, if $|N_x| \geq 1$, then a practice problem suggested in Recitation 4 tells us that $f(x) = (-1)^m \binom{|N_x|-1}{m}$. Therefore, if $|N_x| \geq 1$, then $f(x) \geq 0$ if m is even and $f(x) \leq 0$ if n is odd.

Since $|N_x| = 0$ if and only if $x \notin \bigcup_{i \in [n]} B_i$, the claim follows □

Theorem 4 (Bonferonni inequalities)**.**

$$\left| \Omega \setminus \bigcup_{i \in [n]} B_i \right| \leq \sum_{S \in \binom{[n]}{\leq m}} (-1)^{|S|} |B_S| \qquad \textit{for } m \textit{ even,}$$

$$\left| \Omega \setminus \bigcup_{i \in [n]} B_i \right| \geq \sum_{S \in \binom{[n]}{\leq m}} (-1)^{|S|} |B_S| \qquad \textit{for } m \textit{ odd.}$$

Proof. The proof is identical to the proof of the inclusion-exclusion formula given above. □

Remark. Lemmas 1 and 3 immediately imply the probabilistic versions of inclusion-exclusion and Bonferonni as well. Indeed, since $\mathbf{Pr}[x \in X] = \mathbb{E}\, \mathbf{1}_X$, we can simply use expectations instead of sums, e.g. $\mathbf{Pr}\big[x \in \Omega \setminus \bigcup_{i \in [n]} B_i\big] = \mathbb{E}\, f$.