Definitions. Write the definitions of the following.

(1) $A \subseteq B$

(2) $A = B$

(3) $A \not\subseteq B$

(4) $A \subsetneq B$

(5) $A \setminus B$

(6) $\overline{A}$

(7) $\bigcup_{i\in I} A_i$

(8) $\bigcap_{i\in I} A_i$

(9) $A \times B$

(10) $\mathcal{P}(A)$

(11) $f : A \to B$ is a function

(12) For $X \subseteq A$, $\mathrm{Im}_f(X)$

(13) For $Y \subseteq B$, $\mathrm{PreIm}_f(Y)$

(14) $f : A \to B$ is an injection

(15) $f : A \to B$ is a surjection

(16) $f : A \to B$ is a bijection

(17) $|A| \le |B|$

(18) $|A| = |B|$

(19) $|A| = n$

(20) A is finite

(21) A is countable

(22) A is countably infinite

(23) A is uncountable

(24) $a \mid b$

(25) p is a prime

(26) $\gcd(a, b)$

(27) $a \equiv b \pmod{n}$

Propositional logic.

(1) Show that $p \Rightarrow (q \Rightarrow p)$ is a tautology.

(2) Show that $(\neg p \Rightarrow \text{False}) \Rightarrow p$ is a tautology.

(3) Show that $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

(4) Show that $p \vee q \equiv (\neg p) \Rightarrow q$.

(5) Write the contrapositive: For all $x, y \in \mathbb{R}$, if $x \le y + \epsilon$ for every $\epsilon > 0$, then $x \le y$.

(6) Write the negation: For every $\epsilon > 0$, there is some $\delta > 0$ such that whenever $|x - x_0| < \delta$, it must be the case that $|y - y_0| < \epsilon$.

Sets.

(1) Show that $\overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$.

(2) Show that $A \setminus B = B \setminus A$ if and only if $A = B$.

(3) Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(4) Let $f : A \to B$ and $X, Y \subseteq B$. Show that $\mathrm{PreIm}_f(A \cap B) = \mathrm{PreIm}_f(A) \cap \mathrm{PreIm}_f(B)$.

(5) Show that $\bigcup_{n\in\mathbb{N}} \mathcal{P}([n]) \subseteq \mathcal{P}(\mathbb{N})$. Show that $\mathcal{P}(\mathbb{N}) \not\subseteq \bigcup_{n\in\mathbb{N}} \mathcal{P}([n])$.

Induction.

(1) Let $a_0, a_1, a_2, \ldots$ be a sequence defined by $a_0 = 1$, $a_1 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \ge 2$. Prove that $a_n = 3 \cdot 2^n - 2 \cdot (-1)^n$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) Let $f_0, f_1, \ldots$ be the Fibonacci numbers, i.e. $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \ge 2$. Prove that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{N}$.

(3) If $\varphi \in \mathbb{R}$ has the property that $\varphi^2 = 1 + \varphi$, show that $\varphi^n = \varphi \cdot f_n + f_{n-1}$ for all $n \in \mathbb{N}$.

Properties of the integers.

(1) Let $x, y, z \in \mathbb{Z}$ with $\gcd(x, y) = 1$. Show that if $x \mid z$ and $y \mid z$, then $xy \mid z$.

(2) For $a, b \in \mathbb{Z}$, what numbers can be written as $ax + by$ for some integers x, y?

(3) When does $ax \equiv b \pmod{n}$ have an integer solution for x?

(4) Let $a =$ "$a_n a_{n-1} \cdots a_1 a_0$" where a_i is the ith digit of the decimal form of a. Using modular arithmetic, show that a is even if and only if a_0 is even.

(5) Show that if p is a prime, then $\sqrt{p}$ is irrational.

Jections and cardinality.

(1) Give an example of an injection $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

(2) Give an example of a surjection $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$.

(3) Show that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^{x-1}(2y - 1)$ is a bijection.

(4) Let $f : A \to B$ and $g : B \to C$. Show that if f, g are injections, then $g \circ f$ is an injection.

(5) Show that if f, g are surjections, then $g \circ f$ is a surjection.

(6) Without using the fact that $\mathbb{Q}$ is countable, write $\mathbb{Q}$ as the countable union of countable sets.

(7) Show that if A is uncountable and B is countable, then $A \setminus B$ is uncountable.

(8) Find an example of two uncountable sets A, B where $A \setminus B$ is countably infinite.