Definitions. Write the definitions of the following.

(1) $A \subseteq B$  (10) $\mathcal{P}(A)$  (19) $|A| = n$
(2) $A = B$  (11) $f : A \rightarrow B$ is a function  (20) $A$ is finite
(3) $A \not\subseteq B$  (12) For $X \subseteq A$, $\text{Im}_f(X)$  (21) $A$ is countable
(4) $A \not\subseteq B$  (13) For $Y \subseteq B$, $\text{PreIm}_f(Y)$  (22) $A$ is countably infinite
(5) $A \setminus B$  (14) $f : A \rightarrow B$ is an injection  (23) $A$ is uncountable
(6) $\overline{A}$  (15) $f : A \rightarrow B$ is a surjection  (24) $a \mid b$
(7) $\bigcup_{i \in I} A_i$  (16) $f : A \rightarrow B$ is a bijection  (25) $p$ is a prime
(8) $\bigcap_{i \in I} A_i$  (17) $|A| \leq |B|$  (26) $\gcd(a,b)$
(9) $A \times B$  (18) $|A| = |B|$  (27) $a \equiv b \pmod{n}$

Propositional logic.

(1) Show that $p \Rightarrow (q \Rightarrow p)$ is a tautology.
(2) Show that $(\neg p \Rightarrow \text{False}) \Rightarrow p$ is a tautology.
(3) Show that $\neg(p \lor q) \equiv (\neg p) \land (\neg q)$.
(4) Show that $p \lor q \equiv (\neg p) \Rightarrow q$.
(5) Write the contrapositive: For all $x, y \in \mathbb{R}$, if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$.
(6) Write the negation: For every $\epsilon > 0$, there is some $\delta > 0$ such that whenever $|x - x_0| < \delta$, it must be the case that $|y - y_0| < \epsilon$.

Sets.

(1) Show that $\bigcap_{i \in I} A_i = \bigcup_{i \in I} \overline{A_i}$.
(2) Show that $A \setminus B = B \setminus A$ if and only if $A = B$.
(3) Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(4) Let $f : A \rightarrow B$ and $X, Y \subseteq B$. Show that $\text{PreIm}_f(A \cap B) = \text{PreIm}_f(A) \cap \text{PreIm}_f(B)$.
(5) Show that $\bigcup_{n \in \mathbb{N}} \mathcal{P}([n]) \subseteq \mathcal{P}(\mathbb{N})$. Show that $\mathcal{P}(\mathbb{N}) \not\subseteq \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$.

Induction.

(1) Let $a_0, a_1, a_2, \ldots$ be a sequence defined by $a_0 = 1$, $a_1 = 8$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$. Prove that $a_n = 3 \cdot 2^n - 2 \cdot (-1)^n$ for all $n \in \mathbb{N} \cup \{0\}$.
(2) Let $f_0, f_1, \ldots$ be the Fibonacci numbers, i.e. $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that $\sum_{i=1}^{n} f_i = f_{n+2} - 1$ for all $n \in \mathbb{N}$.
(3) If $\varphi \in \mathbb{R}$ has the property that $\varphi^2 = 1 + \varphi$, show that $\varphi^n = \varphi \cdot f_n + f_{n-1}$ for all $n \in \mathbb{N}$.

Properties of the integers.

(1) Let $x, y, z \in \mathbb{Z}$ with $\gcd(x, y) = 1$. Show that if $x \mid z$ and $y \mid z$, then $xy \mid z$.
(2) For $a, b \in \mathbb{Z}$, what numbers can be written as $ax + by$ for some integers $x, y$?
(3) When does $ax \equiv b \pmod{n}$ have an integer solution for $x$?
(4) Let $a = \ldots a_{i-1}a_i a_0$ where $a_i$ is the $i$th digit of the decimal form of $a$. Using modular arithmetic, show that $a$ is even if and only if $a_0$ is even.
(5) Show that if $p$ is a prime, then $\sqrt{p}$ is irrational.

Injections and cardinality.

(1) Give an example of an injection $f : \mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})$.
(2) Give an example of a surjection $f : \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{N}$.
(3) Show that $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(x, y) = 2^{x-1}(2y - 1)$ is a bijection.
(4) Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Show that if $f, g$ are injections, then $g \circ f$ is an injection.
(5) Show that if $f, g$ are surjections, then $g \circ f$ is a surjection.
(6) Without using the fact that $\mathbb{Q}$ is countable, write $\mathbb{Q}$ as the countable union of countable sets.
(7) Show that if $A$ is uncountable and $B$ is countable, then $A \setminus B$ is uncountable.
(8) Find an example of two uncountable sets $A, B$ where $A \setminus B$ is countably infinite.