Justify all answers!

(18 pts)

(1) [+2] What is the error in the following "proof" that $\mathbb{R}$ is countable?

Proof. Let $F = \{y \in \mathbb{R} : 0 \leq y < 1\}$ and for $y \in F$, define $X_y = \{y + c : c \in \mathbb{Z}\}$. For example, $X_{1/2} = \{\dots, -3/2, -1/2, 1/2, 3/2, \dots\}$. Certainly each X_y is countable as there is a natural bijection with $\mathbb{Z}$. Additionally

$$\mathbb{R} = \bigcup_{y \in F} X_y,$$

as any real number can be written as a sum of its integer part and its fractional part. Thus, $\mathbb{R}$ is the union of countable sets, so $\mathbb{R}$ itself is countable. □

Problem. In class we proved that the *countable* union of countable sets is countable. However, F is an uncountable set, as we showed in class, and it is not necessarily the case that an *uncountable* union of countable sets is countable. In fact, it is easy to show that the uncountable union of nonempty, disjoint sets must be uncountable, so as the X_y's are nonempty and disjoint, it actually *must* be the case that $\mathbb{R}$ is uncountable as F is uncountable. □

(2) In this exercise, we will consider Cartesian products of countable sets.

(a) [+8] Let $X_1, X_2, \dots, X_n$ be countable sets. Prove that $X_1 \times X_2 \times \cdots \times X_n$ is countable. Keep in mind that $A \times B \times C \neq (A \times B) \times C$. (Note: we proved in class that $X_1 \times X_2$ is countable, so you are free to use this fact)

Proof 1. We prove by induction on n.

Base case: If $n = 1$, then X_1 is countable by assumption. If $n = 2$, then $X_1 \times X_2$ is countable as we proved in class.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, $X_1 \times \cdots \times X_{n_0}$ is countable.

Induction step: We must show that $X_1 \times \cdots \times X_{n_0+1}$ is countable. We first notice that $|X_1 \times \cdots \times X_{n_0+1}| = |(X_1 \times \cdots \times X_{n_0}) \times X_{n_0+1}|$ as is illustrated by the natural bijection $(x_1, \dots, x_{n_0+1}) \mapsto \big((x_1, \dots, x_{n_0}), x_{n_0+1}\big)$, so let $A = X_1 \times \cdots \times X_{n_0}$, so $|X_1 \times \cdots \times X_{n_0+1}| = |A \times X_{n_0+1}|$. By the induction hypothesis, A is countable, so as X_{n_0+1} is countable, we find that $A \times X_{n_0+1}$ is countable by the $n = 2$ case. Thus as $|X_1 \times \cdots \times X_{n_0+1}| = |A \times X_{n_0+1}|$, we know that $X_1 \times \cdots \times X_{n_0+1}$ is also countable.

By the principle of mathematical induction, we have shown the claim. □

Proof 2. We prove by induction on n.

Base case: If $n = 1$, then X_1 is countable by assumption.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, $X_1 \times \cdots \times X_{n_0}$ is countable.

Induction step: We must show that $X_1 \times \cdots \times X_{n_0+1}$ is countable. For $y \in X_{n_0+1}$, define the set $A_y = \{(x_1, \dots, x_{n_0}, y) : x_i \in X_i\}$. It is clear that $X_1 \times \cdots \times X_{n_0+1} = \bigcup_{y \in X_{n_0+1}} A_y$. By induction $X_1 \times \cdots \times X_{n_0}$ is countable, so as there is a straightforward bijection between this and A_y for any fixed $y \in X_{n_0+1}$, namely $(x_1, \dots, x_{n_0}) \mapsto (x_1, \dots, x_{n_0}, y)$, we know that A_y is countable. Thus, as X_{n_0+1} is countable, we have written $X_1 \times \cdots \times X_{n_0+1}$ as the countable union of countable sets; thus it must be countable. □

Proof 3. Let $\mathbb{N}^n = \overbrace{\mathbb{N} \times \cdots \times \mathbb{N}}^{n}$. As X_i is countable, there is an injection $f_i : X_i \to \mathbb{N}$. Thus, the function $f : X_1 \times \cdots \times X_n \to \mathbb{N}^n$ defined by $f(x_1, \ldots, x_n) = \big(f_1(x_1), \ldots, f_n(x_n)\big)$ is also an injection. As such, it suffices to show that $\mathbb{N}^n$ is countable.

To do this, we will establish an injection $f : \mathbb{N}^n \to \mathbb{N}$. To do so, let $p_1 = 2$, $p_2 = 3$ and so on. In particular, p_i is the ith prime number. Now, define $f : \mathbb{N}^n \to \mathbb{N}$ by $f(x_1, \ldots, x_n) = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n}$ for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$. Certainly f is well-defined.

Suppose that for some $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{N}^n$, we have $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$; thus $p_1^{x_1} \cdots p_n^{x_n} = p_1^{y_1} \cdots p_n^{y_n}$. By the Fundamental Theorem of Arithmetic, as $p_1, \ldots, p_n$ are distinct primes, we must have $x_1 = y_1$, $x_2 = y_2$, and so on. Thus, $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, so f is an injection. □

(b) [+8] Let $X = \prod_{n \in \mathbb{N}} \{0, 1\}$, in other words, $X = \{(x_1, x_2, x_3, \ldots) : x_n \in \{0, 1\} \text{ for all } n \in \mathbb{N}\}$. Prove that X is uncountable. (Hint: There are many ways to do this. I suggest either modifying the diagonalization proof that $\mathbb{R}$ is uncountable or coming up with a bijection to a set that we already know is uncountable)

Proof 1. Suppose for the sake of contradiction that $f : \mathbb{N} \to X$ is a surjection and let $f(n) = (a_{n1}, a_{n2}, a_{n3}, \ldots)$. Now, define

$$x_n = \begin{cases} 1 & \text{if } a_{nn} = 0; \\ 0 & \text{if } a_{nn} = 1. \end{cases}$$

Certainly $(x_1, x_2, \ldots) \in X$, so there must be some $N \in \mathbb{N}$ with $f(N) = (x_1, x_2, \ldots)$ as f is a surjection. However, by assumption, $f(N) = (a_{N1}, a_{N2}, \ldots)$, so we must have $x_1 = a_{N1}$, $x_2 = a_{N2}$ and so on. In particular, $x_N = a_{NN}$. However, based on how we defined x_N, if $a_{NN} = 0$, then $x_N = 1$ and if $a_{NN} = 1$, then $x_N = 0$, so $x_N \neq a_{NN}$; a contradiction! □

Proof 2. We will show that $|X| = |\mathcal{P}(\mathbb{N})|$, so that X is uncountable as we have shown that $\mathcal{P}(\mathbb{N})$ is uncountable. To do this, we will provide an explicit bijection. This bijection will be similar to the one we established in class to show $|\mathcal{P}([n])| = 2^n$.

For a string $\mathbf{x} = (x_1, x_2, \ldots) \in X$, define $S_{\mathbf{x}} = \{n \in \mathbb{N} : x_n = 1\}$. Now let $f : X \to \mathcal{P}(\mathbb{N})$ be defined by $f(\mathbf{x}) = S_{\mathbf{x}}$. Certainly f is a function with codomain $\mathcal{P}(\mathbb{N})$ as $S_{\mathbf{x}} \subseteq \mathbb{N}$ for all $\mathbf{x} \in X$.

First we argue that f is an injection. Let $\mathbf{x}, \mathbf{x}' \in X$ where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{x}' = (x_1', x_2', \ldots)$ and suppose $\mathbf{x} \neq \mathbf{x}'$. Thus, there is some $n \in \mathbb{N}$ for which $x_n \neq x_n'$. If $x_n = 1$, then $n \in S_{\mathbf{x}}$ but $n \notin S_{\mathbf{x}'}$, so $f(\mathbf{x}) \neq f(\mathbf{x}')$. The same conclusion follows if $x_n' = 1$. Thus, f is an injection.

To argue that f is a surjection, let $S \in \mathcal{P}(\mathbb{N})$. Now define $\mathbf{s} = (s_1, s_2, \ldots)$ where $s_n = 1$ if $n \in S$ and $s_n = 0$ if $n \notin S$. Certainly $\mathbf{s} \in X$ and $f(\mathbf{s}) = S$, so f is a surjection.

As such, f is a bijection, so $|X| = |\mathcal{P}(\mathbb{N})|$, so X is uncountable.

In fact, a very similar bijection can be established to show that for any set Y, $\left|\prod_{y \in Y} \{0, 1\}\right| = |\mathcal{P}(Y)|$. □

(Rewrite) [+2] Do there exist uncountable sets A, B such that $A \setminus B$ is countably infinite?

Proof. Yes. Consider $A = \mathbb{R}$ and $B = \mathbb{R} \setminus \mathbb{N}$. As $\mathbb{R}$ is uncountable and $\mathbb{N}$ is countable, we know that A and B are both uncountable. However, $A \setminus B = \mathbb{N}$, which is countably infinite. □