Justify all answers!

(18 pts)

(1) [+2] What is the error in the following "proof" that $\mathbb{R}$ is countable?

Proof. Let $F = \{y \in \mathbb{R} : 0 \leq y < 1\}$ and for $y \in F$, define $X_y = \{y + c : c \in \mathbb{Z}\}$. For example, $X_{1/2} = \{\ldots, -3/2, -1/2, 1/2, 3/2, \ldots\}$. Certainly each X_y is countable as there is a natural bijection with $\mathbb{Z}$. Additionally

$$\mathbb{R} = \bigcup_{y \in F} X_y,$$

as any real number can be written as a sum of its integer part and its fractional part. Thus, $\mathbb{R}$ is the union of countable sets, so $\mathbb{R}$ itself is countable. □

(2) In this exercise, we will consider Cartesian products of countable sets.

(a) [+8] Let $X_1, X_2, \ldots, X_n$ be countable sets. Prove that $X_1 \times X_2 \times \cdots \times X_n$ is countable. Keep in mind that $A \times B \times C \neq (A \times B) \times C$. (Note: we proved in class that $X_1 \times X_2$ is countable, so you are free to use this fact)

(b) [+8] Let $X = \prod_{n \in \mathbb{N}} \{0, 1\}$, in other words, $X = \{(x_1, x_2, x_3, \ldots) : x_n \in \{0, 1\} \text{ for all } n \in \mathbb{N}\}$. Prove that X is uncountable. (Hint: There are many ways to do this. I suggest either modifying the diagonalization proof that $\mathbb{R}$ is uncountable or coming up with a bijection to a set that we already know is uncountable)