Throughout this assignment, you may freely use the identity $(x-1)\sum_{i=0}^{n} x^i = x^{n+1}-1$, which holds for all $x \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N} \cup \{0\}$, which we proved in class.

Justify all answers!

(22 pts)

(1) [+8] Let $a, n \in \mathbb{N}$ with $n \geq 2$. Prove that if $a^n - 1$ is a prime, then $a = 2$ and n is a prime.

(2) For $n \in \mathbb{N}$, define

$$\sigma(n) := \sum_{d|n} d.$$

For example, $\sigma(12) = 1+2+3+4+6+12$. We say that n is *perfect* if $\sigma(n) = 2n$. For example, 6 is perfect as $\sigma(6) = 1+2+3+6 = 12$.

- (a) [+2] Let p be a prime and $k \in \mathbb{N}$. Show that $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$.
- (b) [+6] Let p be a prime and $n \in \mathbb{N}$ with $p \nmid n$. Prove that $\sigma(pn) = \sigma(p)\sigma(n)$.
- (c) [+6] Let $n \in \mathbb{N}$ and suppose that $2^n - 1$ is prime. Show that $2^{n-1}(2^n - 1)$ is perfect.
- (d) **Bonus.** [$+\infty$] Prove or disprove: There are no odd perfect numbers.

(3) **Bonus.** [+3] Let p_n be the nth prime, e.g. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc. Prove that $p_n \leq 2^{2^{n-1}}$.[1]

[1] In fact, it can be shown through a similar idea that $p_n \leq 2^n$, but we have not developed all of the tools to do this. Through much more complicated techniques, it can actually be shown that $p_n = \big(1 + c(n)\big) n \log n$ where $c(n) \to 0$ as $n \to \infty$.