In this assignment, we will explore the Fibonacci numbers, which are defined as follows:

$$f_0 = 0,$$
$$f_1 = 1,$$
$$f_n = f_{n-1} + f_{n-2} \text{ , for } n \geq 2.$$

(22 pts)

(1) [+1] What is f_{11}?

Proof. $f_{11} = 89$ □

(2) [+7] For all $n \in \mathbb{N} \cup \{0\}$, prove that $f_n < 2^n$.

Proof. We prove by strong induction. We will need two base cases.

Base case: For $n = 0$, $f_0 = 0 < 1 = 2^0$. For $n = 1$, $f_1 = 1 < 2 = 2^1$.

Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, if $m \in \mathbb{N} \cup \{0\}$ has $m < n_0$, then $f_m < 2^m$.

Induction step: Assuming the induction hypothesis, prove that $f_{n_0} < 2^{n_0}$.

As we have already shown the claim for $n = 0, 1$, we may suppose that $n_0 \geq 2$. In this case, we may use the fact that $f_{n_0} = f_{n_0-1} + f_{n_0-2}$. Appealing to the strong inductive hypothesis,

$$f_{n_0} = f_{n_0-1} + f_{n_0-2} < 2^{n_0-1} + 2^{n_0-2} = 2^{n_0}\left(\frac{1}{2} + \frac{1}{4}\right) < 2^{n_0}.$$

Thus, by strong induction, we have shown that $f_n < 2^n$ for all $n \in \mathbb{N} \cup \{0\}$. □

(3) [+7] For all $n \in \mathbb{N}$, prove that

$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}.$$

Proof. We prove by weak induction.

Base case: For $n = 1$, we have

$$\sum_{k=1}^{1} f_k^2 = f_1^2 = 1 = f_1 f_2,$$

as $f_1 = f_2 = 1$.

Induction hypothesis: For some $n_0 \in \mathbb{N}$ it is true that $\sum_{k=1}^{n_0} f_k^2 = f_{n_0} f_{n_0+1}$.

Induction step: Assuming the induction hypothesis, prove that $\sum_{k=1}^{n_0+1} f_k^2 = f_{n_0+1} f_{n_0+2}$.

We calculate

$$\begin{aligned}
\sum_{k=1}^{n_0+1} f_k^2 &= f_{n_0+1}^2 + \sum_{k=1}^{n_0} f_k^2 \\
&= f_{n_0+1}^2 + f_{n_0} f_{n_0+1} && \text{(by the induction hypothesis)} \\
&= f_{n_0+1}\big(f_{n_0+1} + f_{n_0}\big) \\
&= f_{n_0+1} f_{n_0+2}.
\end{aligned}$$

Thus, by the principle of mathematical induction, we have shown that the claim holds for all $n \in \mathbb{N}$. □

(4) [+7] For all $n \in \mathbb{N}$, prove that

$$\sum_{k=1}^{n} k \cdot f_k = n \cdot f_{n+2} - f_{n+3} + 2.$$

Proof. We prove by induction on n.

Base case: For $n = 1$, $\sum_{k=1}^{1} k \cdot f_k = 1 = 2 - 3 + 2 = 1 \cdot f_{1+2} - f_{1+3} + 2$.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, we have $\sum_{k=1}^{n_0} k \cdot f_k = n_0 f_{n_0+2} - f_{n_0+3} + 2$.

Induction step: Assuming the induction hypothesis, we must show that $\sum_{k=1}^{n_0+1} k \cdot f_k = (n_0+1) f_{n_0+3} - f_{n_0+4} + 2$.

We calculate

$$\begin{aligned}
\sum_{k=1}^{n_0+1} k \cdot f_k &= \sum_{k=1}^{n_0} k \cdot f_k + (n_0 + 1) f_{n_0+1} \\
&= \big(n_0 f_{n_0+2} - f_{n_0+3} + 2\big) + (n_0 + 1) f_{n_0+1} && \text{(By the IH)} \\
&= (n_0 + 1)\big(f_{n_0+2} + f_{n_0+1}\big) - f_{n_0+2} - f_{n_0+3} + 2 \\
&= (n_0 + 1) f_{n_0+3} - f_{n_0+4} + 2.
\end{aligned}$$

Thus, by the principle of mathematical induction, the claim is true for all $n \in \mathbb{N}$. □

(5) **Bonus.** [+4] Prove that every natural number can be written as the sum of one or more *distinct* Fibonacci numbers. That is, for every $n \in \mathbb{N}$, there are *distinct* $k_1, \ldots, k_\ell$ with $n = f_{k_1} + f_{k_2} + \cdots + f_{k_\ell}$. For example,

$$5 = 5 = f_5, \quad 6 = 5 + 1 = f_5 + f_1, \quad 10 = 8 + 2 = f_6 + f_3, \quad 12 = 8 + 3 + 1 = f_6 + f_5 + f_1.$$

Proof. We apply strong induction here.

Base case: For $n = 1$, $1 = f_1$.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, if $m < n_0$ and $m \in \mathbb{N}$, then m can be written as the sum of distinct Fibonacci numbers.

Induction step: Assuming the induction hypothesis, we must prove that n_0 can be written as the sum of distinct Fibonacci numbers.

Let K be such that f_K is the largest Fibonacci number less than or equal to n_0. If $n_0 = f_K$, then we are done. Otherwise, set $N = n_0 - f_K$. We notice that $1 \leq N < n_0$, so by the induction hypothesis, we may write $N = f_{k_1} + \cdots + f_{k_\ell}$ for some $k_1 > \cdots > k_\ell$. As such, $n_0 = f_{k_1} + \cdots + f_{k_\ell} + f_K$. We now must argue that $K \notin \{k_1, \ldots, k_\ell\}$, so that all of these Fibonacci numbers are distinct. To see this, we rely on the fact that f_K was the largest Fibonacci number less than or equal to n_0; thus $f_{K+1} > n_0$. As such, $N = n_0 - f_K < f_{K+1} - f_K = f_{K-1}$, so $f_K \geq f_{K-1} > N = f_{k_1} + \cdots + f_{k_\ell}$. In particular, $K > k_1 > \cdots > k_\ell$, so $K \notin \{k_1, \ldots, k_\ell\}$. □

(Rewrite) [+3] Let φ be a root of the equation $X^2 - X = 1$, that is to say $\varphi^2 = 1 + \varphi$. Prove that $\varphi^n = \varphi \cdot f_n + f_{n-1}$ for all $n \in \mathbb{N}$. (Note: it is not necessary to calculate what φ actually is.)

Proof. We prove by induction on n.

Base case: For $n = 1$, $\varphi \cdot f_1 + f_0 = \varphi \cdot 1 + 0 = \varphi^1$.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, $\varphi^{n_0} = \varphi \cdot f_{n_0} + f_{n_0-1}$.

Induction step: We must show that $\varphi^{n_0+1} = \varphi \cdot f_{n_0+1} + f_{n_0}$. We calculate

$$\begin{aligned}
\varphi^{n_0+1} &= \varphi \cdot \varphi^{n_0} \\
&= \varphi\big(\varphi \cdot f_{n_0} + f_{n_0-1}\big) && \text{(By the induction hypothesis)} \\
&= \varphi^2 \cdot f_{n_0} + \varphi \cdot f_{n_0-1} \\
&= \varphi\big(f_{n_0} + f_{n_0+1}\big) + f_{n_0} && \text{(As } \varphi^2 = 1 + \varphi\text{)} \\
&= \varphi \cdot f_{n_0+1} + f_{n_0}.
\end{aligned}$$

Thus, by the principle of mathematical induction, the claim holds for all $n \in \mathbb{N}$. $\square$