In this assignment, we will explore a few fun properties of the Fibonacci numbers. The Fibonacci numbers are defined by the recursive relation:

$$f_0 = 0,$$
$$f_1 = 1,$$
$$f_n = f_{n-1} + f_{n-2} \text{ , for } n \geq 2.$$

(22 pts)

(1) [+1] What is f_{11}?

(2) [+7] For all $n \in \mathbb{N} \cup \{0\}$, prove that $f_n < 2^n$.

(3) [+7] For all $n \in \mathbb{N}$, prove that

$$\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}.$$

(4) [+7] For all $n \in \mathbb{N}$, prove that

$$\sum_{k=1}^{n} k \cdot f_k = n \cdot f_{n+2} - f_{n+3} + 2.$$

(5) **Bonus.** [+4] Prove that every natural number can be written as the sum of one or more *distinct* Fibonacci numbers. That is, for every $n \in \mathbb{N}$, there are *distinct* $k_1, \ldots, k_\ell$ with $n = f_{k_1} + f_{k_2} + \cdots + f_{k_\ell}$. For example,

$$5 = 5 = f_5, \quad 6 = 5 + 1 = f_5 + f_1, \quad 10 = 8 + 2 = f_6 + f_3, \quad 12 = 8 + 3 + 1 = f_6 + f_5 + f_1.$$