Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is *increasing* if $f(x) \geq f(y)$ whenever $x \geq y$. Similarly f is *decreasing* if $f(x) \leq f(y)$ whenever $x \geq y$.

(25 pts)

(1) Let $f, g : \mathbb{R} \to \mathbb{R}$. Prove or disprove the following statements.

(a) [+3] If f, g are increasing, then $f \circ g$ is increasing.

Proof. Let $x \leq y$, then as g is increasing $g(x) \leq g(y)$. Also, as f is increasing, $(f \circ g)(x) = f(g(x)) \leq f(g(y)) = (f \circ g)(y)$, so $f \circ g$ is increasing. □

(b) [+3] If f is increasing and g is decreasing, then $f \circ g$ is decreasing.

Proof. Let $x \leq y$, then as g is decreasing $g(x) \geq g(y)$. As f is increasing, $(f \circ g)(x) = f(g(x)) \geq f(g(y)) = (f \circ g)(y)$, so $f \circ g$ is decreasing. □

(c) [+3] If f is increasing and g is decreasing, then $f - g$ is increasing.

Proof. Let $x \leq y$, then as f is increasing $f(x) \leq f(y)$ and as g is decreasing $g(x) \geq g(y)$, so $-g(x) \leq -g(y)$. Thus, $(f-g)(x) = f(x) - g(x) \leq f(y) - g(y) = (f-g)(y)$, so $f - g$ is increasing. □

(d) [+3] If f, g are increasing, then fg is increasing.

Proof. This is false. Let $f(x) = g(x) = x$, so certainly both f and g are increasing. However, $(fg)(x) = x^2$ which is not increasing as witnessed by $(fg)(-2) = 4 > 1 = (fg)(-1)$ even though $-2 < -1$. □

(2) For sets A, B, define the *symmetric difference* to be $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

(a) [+5] Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Proof. ($\subseteq$) Let $x \in A \triangle B$ be arbitrary. By definition, this means that $x \in A \setminus B$ or $x \in B \setminus A$. First suppose $x \in A \setminus B$, then certainly $x \in A \subseteq A \cup B$, so $x \in A \cup B$. Additionally, $x \notin B$, so $x \notin A \cap B$. Thus $x \in (A \cup B) \setminus (A \cap B)$. The same argument up to switching the roles of A and B works if $x \in B \setminus A$, so we have shown that $A \triangle B \subseteq (A \cup B) \setminus (A \cap B)$.

($\supseteq$) Let $x \in (A \cup B) \setminus (A \cap B)$ be arbitrary. Therefore, $x \in A \cup B$ and $x \notin A \cap B$. First suppose $x \in A$. As $x \notin A \cap B$, this means that $x \notin B$, so $x \in A \setminus B$, so $x \in A \triangle B$. On the other hand, if $x \in B$, then as $x \notin A \cap B$, we have $x \notin A$. Thus, $x \in B \setminus A$, so $x \in A \triangle B$. Thus, we have $(A \cup B) \setminus (A \cap B) \subseteq A \triangle B$. □

(3) [+8] Which of the following is true? Prove your claim.

(a) $\mathcal{P}(\mathbb{N}) \subsetneq \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$

(b) $\mathcal{P}(\mathbb{N}) \supsetneq \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$

(c) $\mathcal{P}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$.

Proof. (b) is true. We first prove the containment and then show it is proper. Let $S \in \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$ be arbitrary. By the definition of union, there is some $n \in \mathbb{N}$ for which $S \in \mathcal{P}([n])$, so $S \subseteq [n]$. As $[n] \subseteq \mathbb{N}$, we also have $S \subseteq \mathbb{N}$, so $S \in \mathcal{P}(\mathbb{N})$. As $S \in \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$ was arbitrary, $\bigcup_{n \in \mathbb{N}} \mathcal{P}([n]) \subseteq \mathcal{P}(\mathbb{N})$.

However, we do not have equality. Consider the set $\mathbb{N}$. Certainly $\mathbb{N} \in \mathcal{P}(\mathbb{N})$; however, suppose $\mathbb{N} \in \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$. Then there must be some n for which $\mathbb{N} \in \mathcal{P}([n])$, so $\mathbb{N} \subseteq [n]$. However, $n+1 \in \mathbb{N}$ but $n+1 \notin [n]$; a contradiction. Thus, $\mathbb{N} \notin \bigcup_{n \in \mathbb{N}} \mathcal{P}([n])$, so $\bigcup_{n \in \mathbb{N}} \mathcal{P}([n]) \subsetneq \mathbb{N}$. □

(Rewrite) [+3] Prove that if $A \triangle B = \varnothing$, then $A = B$.

Proof. Suppose that $A \triangle B = \varnothing$. As $A \setminus B, B \setminus A \subseteq A \triangle B$, we must have $A \setminus B = \varnothing$ and $B \setminus A = \varnothing$. The former implies that $A \subseteq B$ and the latter implies that $B \subseteq A$. Thus $A = B$. $\square$