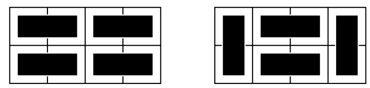
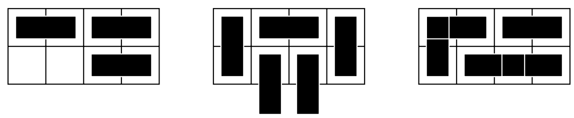
In this assignment, we will explore chessboard tilings. A domino covers exactly two squares of a chessboard that are either horizontally or vertically adjacent. A proper tiling of a board is one where each square is covered *exactly* one domino. For example the following are proper tilings of the 2×4 board with dominoes



whereas these are not



Justify all answers! If a tiling is possible, exhibit one or give a process which will create one. If a tiling is impossible, you must prove this.

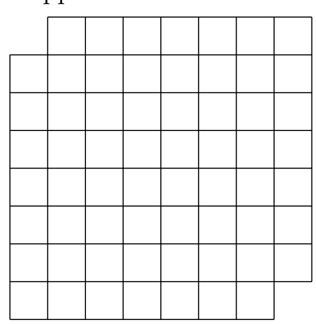
(27 pts)

(1) [+4] For what values of m and n does an $m \times n$ board have a domino tiling?

Proof. An $m \times n$ board has a domino tiling if and only if mn is even. First off, suppose an $m \times n$ board has a domino tiling, then as a domino covers two squares and each square of the board is covered exactly once, we see that the board must have an even number of squares. In other words mn must be even.

On the other hand, suppose mn is even, then m or n is even. Without loss of generality m is even (in other words, the case where m is even verses the case where n is even are essentially identical). Certainly we can tile an $m \times 1$ board as m is even by laying $m/2$ dominoes end-to-end. Repeating this tiling on every column of the $m \times n$ board yields a domino tiling. □

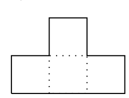
(2) [+8] Consider the 8×8 board with two opposite corners removed:



Does this board have a domino tiling?

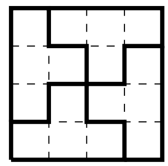
Proof. No, the board does not have a domino tiling. Consider coloring the squares white and black in the natural chessboard fashion. Then every domino covers exactly one white and one black square. Thus, if it were to have a domino tiling, there must be an equal number of white and black squares. However, the opposite corners of this board are the same color, so this is not the case. □

(3) Consider the following object called a tetromino:



(a) [+5] Does the 8×8 board have a tetromino tiling? (Note: the tetromino can be rotated, and we are talking about the standard 8×8 board and not the modified one in the previous question.)

Proof. Yes, the 8×8 board does have a tetromino tiling. Consider the following tetromino tiling of the 4×4 board:



Now, for any positive integer n, we can tile the $4n \times 4n$ board by placing n disjoint copies of the above tiling. Therefore the 8×8 board has a tetromino tiling by taking $n = 2$. □

(b) [+10] How about the 10×10 board?

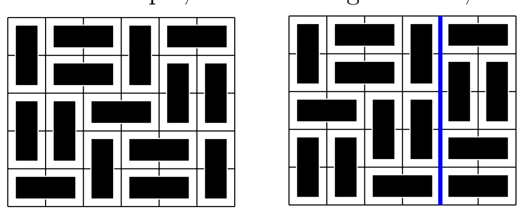
Proof. No, the 10×10 board does not have a tetromino tiling. Suppose it did, then there are two "types" of tetrominoes used. Namely, if we again color the board as a chessboard, there are the tetrominoes that cover one black square and three white and the tetrominoes that cover one white square and three black. Suppose there are b of the first kind and w of the second. As there are 50 black and 50 white squares in the 10×10 board, we must have

$$b + 3w = 50$$
$$3b + w = 50.$$

Solving this system of equations yields $b = w = 25/2$, which is not an integer. Thus, the 10×10 board cannot have a tetromino tiling.

Notice that this same argument show that an $n \times n$ board with $4 \nmid n$ cannot have a tetromino tiling. This, along with (3a) shows that an $n \times n$ board has a tetromino tiling if and only if $4 \mid n$. □

(4) **Bonus.** [+4] A domino tiling is said to be *stable* if every horizontal and vertical line drawn through the board must cross some domino. For example, the first tiling is stable, while the second is not.



Does the 6×6 board have a stable domino tiling?

Proof. No, the 6×6 board does not have a stable domino tiling. To see this, we will first prove the following claim: for positive integers m, n, in any domino tiling of the $2m \times 2n$ board, every horizontal and vertical line must be crossed by an even number of dominoes. We will only prove this for the vertical lines, and the claim for the horizontal lines is equivalent as we could simply reflect the board.

Say that a domino is of type (i, j) if it is in columns i and j. Note that if a domino is of type (i, j) then $|i - j| \leq 1$, but this is not very important. Look at the first column of the $2m \times 2n$ board. If there were an odd number of $(1, 2)$ dominoes, then the number of squares in the first column that must be covered by $(1, 1)$ dominoes is odd as there are $2m$ rows. However, each $(1, 1)$ domino covers two squares in the first column, and as we have a domino tiling, this means that the number of squares in the first column covered by $(1, 1)$ dominoes must be even; a contradiction. Therefore, the number of squares in the first column covered by $(1, 2)$ dominoes must be even. Now, consider looking at the second column. We know that an even number of squares are already covered by $(1, 2)$ dominoes, so there are an even number of squares in the second column that must be covered by $(2,2)$ and $(2,3)$ dominoes. By the same argument as earlier, this means that there must be an even number of $(2,3)$ dominoes. Repeating this argument across the board, we know that there must be an even number of $(i,i+1)$ dominoes for each $1 \leq i \leq 2n-1$. In particular, each vertical line must be crossed by an even number of dominoes.

Now, suppose a $2m \times 2n$ board had a stable domino tiling. As this is a domino tiling, each horizontal and vertical line must be crossed by an even number of dominoes, as we just proved. However, as the tiling is stable, each line must be crossed at least once, so in fact each line must be crossed at least twice. Thus, as a domino can cross exactly one line, the number of dominoes necessary in such a tiling is at least $2(2m-1)+2(2n-1)=4(m+n-1)$. However, as the number of squares is $4mn$, we have only $2mn$ dominoes, so it must be the case that

$$2mn \geq 4(m+n-1).$$

In the case of the 6×6 board (i.e. $m=n=3$), if there were to be a stable domino tiling, then it must be the case that $18 \geq 20$, which is not true. Thus, the 6×6 board cannot have such a tiling.

Interestingly, the following is true: an $m \times n$ board has a stable domino tiling if and only if mn is even, $m,n \geq 5$ and $(m,n) \neq (6,6)$. In other words, the 6×6 board is quite special! $\square$