Justify all answers!

(8 pts)

(1) [+8] Let $x \in \mathbb{R}$ have the property that $x + \frac{1}{x} \in \mathbb{Z}$. Prove that $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. (Hint: first prove it for $n \in \mathbb{N} \cup \{0\}$ and then extend to all of $\mathbb{Z}$)

Proof. For simplicity, set $a_n = x^n + \frac{1}{x^n}$.

We first prove that the claim holds for all $n \in \mathbb{N} \cup \{0\}$ by strong induction on n.

Base Cases: For $n = 0$, $a_0 = x^0 + \frac{1}{x^0} = 2 \in \mathbb{Z}$. Also, for $n = 1$, $a_1 = x + \frac{1}{x} \in \mathbb{Z}$ by assumption.

Induction hypothesis: Suppose that for some $n_0 \in \mathbb{N} \cup \{0\}$, $a_m \in \mathbb{Z}$ for all $m < n_0$ with $m \in \mathbb{N} \cup \{0\}$.

Induction step: Assuming the induction hypothesis, we must prove that $a_{n_0} \in \mathbb{Z}$. Due to the base cases, we may suppose $n_0 \geq 2$, so, in particular, the induction hypothesis applies to a_{n_0-1} and a_{n_0-2}. We proceed by "completing the square" to calculate,

$$\begin{aligned} a_{n_0} &= x^{n_0} + \frac{1}{x^{n_0}} \\ &= \left(x + \frac{1}{x}\right)\left(x^{n_0-1} + \frac{1}{x^{n_0-1}}\right) - \left(x^{n_0-2} + \frac{1}{x^{n_0-2}}\right) \\ &= a_1 a_{n_0-1} - a_{n_0-2}. \end{aligned}$$

By the induction hypothesis, $a_1, a_{n_0-1}, a_{n_0-2} \in \mathbb{Z}$, so as a_{n_0} is a product and sum of integers, we also have $a_{n_0} \in \mathbb{Z}$.

Thus, by the principle of mathematical induction, $a_n \in \mathbb{Z}$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, let $n \in \mathbb{Z}$ with $n \leq -1$ and notice that

$$a_n = x^n + \frac{1}{x^n} = x^{-n} + \frac{1}{x^{-n}} = a_{-n}.$$

As $-n \in \mathbb{N}$, we have $a_{-n} \in \mathbb{Z}$, so $a_n \in \mathbb{Z}$. As such, $a_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. □

(Rewrite) [+3] Let $a_0, a_1, a_2, \ldots$ be a sequence defined by $a_0 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$. Prove that $a_n = 2^n - 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. We prove by induction on n

Base case: For $n = 0$, $a_0 = 0 = 2^0 - 1$.

Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, we have $a_{n_0} = 2^{n_0} - 1$.

Induction step: We must show that $a_{n_0+1} = 2^{n_0+1} - 1$. To do this, we appeal to the recurrence relation and calculate

$$\begin{aligned} a_{n_0+1} &= 2a_{n_0} + 1 \\ &= 2\big(2^{n_0} - 1\big) + 1 && \text{(By the induction hypothesis)} \\ &= 2^{n_0+1} - 1. \end{aligned}$$

Thus, by induction, we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N} \cup \{0\}$. □