Recall that for $a_1, \ldots, a_n \in \mathbb{R}$, $\prod_{i=1}^{n} a_i = a_1 a_2 \cdots a_n$.

Justify all answers!

(8 pts)

(1) [+8]Prove that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in [0,1]$,

$$\prod_{i=1}^{n}(1-x_i) \geq 1 - \sum_{i=1}^{n} x_i.$$

Proof. We prove by induction on n.

Base case: For $n = 1$, $\prod_{i=1}^{1}(1-x_i) = 1 - x_i = 1 - \sum_{i=1}^{1} x_i$.

Induction hypothesis: Suppose for some $n_0 \in \mathbb{N}$, for any $x_1, \ldots, x_{n_0} \in [0,1]$, we have $\prod_{i=1}^{n_0}(1-x_i) \geq 1 - \sum_{i=1}^{n_0} x_i$.

Induction step: Let $x_1, \ldots, x_{n_0+1} \in [0,1]$ be arbitrary. Assuming the induction hypothesis, we must show that $\prod_{i=1}^{n_0+1}(1-x_i) \geq 1 - \sum_{i=1}^{n_0+1} x_i$. We now calculate

$$\begin{aligned}
\prod_{i=1}^{n_0+1}(1-x_i) &= \left(\prod_{i=1}^{n_0}(1-x_i)\right)(1-x_{n_0+1}) \\
&\geq \left(1 - \sum_{i=1}^{n_0} x_i\right)(1-x_{n_0+1}) && \text{(By the IH and the fact that } (1-x_{n_0+1}) \geq 0) \\
&= 1 - \sum_{i=1}^{n_0} x_i - x_{n_0+1} + x_{n_0+1}\sum_{i=1}^{n_0} x_i \\
&\geq 1 - \sum_{i=1}^{n_0+1} x_i && \text{(as } x_i \geq 0 \text{ for all } i \in [n_0+1]).
\end{aligned}$$

Thus, by the principle of mathematical induction, we have proved the claim. □

(Rewrite) [+2] Give a counterexample to the claim you were asked to prove in the homework if $x_1, \ldots, x_n$ are allowed to be any real numbers and not required to be in $[0,1]$. That is, for some $n \in \mathbb{N}$, find real numbers $x_1, \ldots, x_n$, for which $\prod_{i=1}^{n}(1-x_i) < 1 - \sum_{i=1}^{n} x_i$.

Proof. Consider $x_1 = -1$ and $x_2 = 2$. Then $(1-x_1)(1-x_2) = -2$ while $1 - (-1+2) = 0$. □