Justify all answers!

## (6 pts)

(1) [+3] What is wrong with the following proof that all integers are equal?

**Claim.** For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{Z}$ it must be the case that $a_1 = \cdots = a_n$.

*Proof.* Base case: For $n = 1$, and any $a_1 \in \mathbb{Z}$, $a_1 = a_1$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, and any $a_1, \ldots, a_{n_0} \in \mathbb{Z}$, it must be the case that $a_1 = \cdots = a_{n_0}$.

Induction step: Given the induction hypothesis, prove that for any $a_1, \ldots, a_{n_0+1} \in \mathbb{Z}$, we have $a_1 = \cdots = a_{n_0+1}$.

Let $a_1, \ldots, a_{n_0+1} \in \mathbb{Z}$ be arbitrary. First consider $a_1, \ldots, a_{n_0}$. As this is a set of $n_0$ integers, by the induction hypothesis, $a_1 = \cdots = a_{n_0}$. Also, $a_2, \ldots, a_{n_0+1}$ is a set of $n_0$ integers, so $a_2 = \cdots = a_{n_0+1}$. By transitivity, we have, $a_1 = a_2 = \cdots = a_{n_0} = a_{n_0+1}$.

Thus, as the statement is true for $n = 1$ and we have shown that if the statement is true for $n = n_0$, then it is true for $n = n_0 + 1$, by the principle of mathematical induction, all integers are equal. □

*Problem:* The problem here is similar to the "all horses are brown" example. Consider what happens if $n_0 = 1$, so $n_0 + 1 = 2$. We have two integers $a_1, a_2$. The induction hypothesis allows us to conclude that $a_1 = a_1$ and $a_2 = a_2$, but does not allow us to conclude that $a_1 = a_2$. □

(2) [+3] What is wrong with the following proof that all natural numbers are equal?

**Claim.** For any $n \in \mathbb{N}$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n$, then $x = y$.

*Proof.* Base Case: For $n = 1$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = 1$, then it must be the case that $x = 1 = y$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n_0$, then $x = y$.

Induction step: Given the induction hypothesis, prove that if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n_0 + 1$, then $x = y$.

Take any $x, y \in \mathbb{N}$ with $\max\{x, y\} = n_0 + 1$, then it is the case that $\max\{x - 1, y - 1\} = n_0$. By the induction hypothesis, we know that $x - 1 = y - 1$, so $x = y$.

Thus, as the statement is true for $n = 1$ and we have show that if the statement is true for $n = n_0$, then it is true for $n = n_0 + 1$, by the principle of mathematical induction, all natural numbers are equal. □

*Problem:* The problem here is that in the induction step, we should not be allowed to apply the induction hypothesis as we may go out of range. For example, consider what happens if $x = 1$ and $y = n_0 + 1$. Certainly $\max\{x, y\} = n_0 + 1$ and $\max\{x - 1, y - 1\} = n_0$; however, $x - 1 = 0 \notin \mathbb{N}$, so we are not allowed to apply the induction hypothesis as it only posits any truth about the statement when $x - 1, y - 1 \in \mathbb{N}$. □

(Rewrite) [+3] What is wrong with the following proof of the fact that

$$\sqrt{1 + \sqrt{1 + 2 \cdot \sqrt{1 + 3 \cdot \sqrt{1 + 4 \cdot \sqrt{1 + 5 \cdot \sqrt{\cdots}}}}}} = 2?$$

Interestingly enough, the above identity is actually true. Additionally, there are two errors in the following proof; it is enough to find just one.

**Claim.** For any $n \in \mathbb{N} \cup \{0\}$,

$$\sqrt{1 + n \cdot \sqrt{1 + (n+1) \cdot \sqrt{1 + (n+2) \cdot \sqrt{1 + (n+3) \cdot \sqrt{1 + (n+4) \cdot \sqrt{\cdots}}}}}} = n + 1.$$

*Proof.* We prove by induction on $n$. Define

$$f(n) = \sqrt{1 + n \cdot \sqrt{1 + (n+1) \cdot \sqrt{1 + (n+2) \cdot \sqrt{1 + (n+3) \cdot \sqrt{1 + (n+4) \cdot \sqrt{\cdots}}}}}}$$

for easier reference. Notice that $f(n) = \sqrt{1 + n \cdot f(n+1)}$.

Base case: If $n = 0$, then $f(n) = 1$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, $f(n_0) = n_0 + 1$.

Induction step: Assuming $f(n_0) = n_0 + 1$, deduce that $f(n_0 + 1) = n_0 + 2$.

We know that $f(n_0) = n_0 + 1$, so squaring both sides gives

$$\begin{aligned} f(n_0)^2 &= (n_0 + 1)^2 \\ \Rightarrow 1 + n_0 \cdot f(n_0 + 1) &= n_0^2 + 2n_0 + 1 \\ \Rightarrow n_0 \cdot f(n_0 + 1) &= n_0(n_0 + 2) \\ \Rightarrow f(n_0 + 1) &= n_0 + 2. \end{aligned}$$

Thus, by the principle of mathematical induction, $f(n) = n + 1$ for all $n \in \mathbb{N} \cup \{0\}$. Setting $n = 1$ yields the original identity. □

*Problem.* There are two problems with this "proof." The first is kind of a meta-problem. It does not need to be the case that this infinite nested radical even exists; a different argument is necessary just to justify calling it a number.

The problem with the induction follows in the last line; when going from $n_0 \cdot f(n_0 + 1) = n_0(n_0 + 2)$, to $f(n_0 + 1) = n_0 + 2$, we divided by $n_0$. However, when $n_0 = 0$, we are not allowed to do that! Note that this occurs when trying to prove $f(1) = 2$. □