(Rewrite) [+3] What is wrong with the following proof of the fact that

$$\sqrt{1+\sqrt{1+2\cdot\sqrt{1+3\cdot\sqrt{1+4\cdot\sqrt{1+5\cdot\sqrt{\cdots}}}}}} = 2?$$

Interestingly enough, the above identity is actually true. Additionally, there are two errors in the following proof; it is enough to find just one.

**Claim.** For any $n \in \mathbb{N} \cup \{0\}$,

$$\sqrt{1+n\cdot\sqrt{1+(n+1)\cdot\sqrt{1+(n+2)\cdot\sqrt{1+(n+3)\cdot\sqrt{1+(n+4)\cdot\sqrt{\cdots}}}}}} = n+1.$$

*Proof.* We prove by induction on $n$. Define

$$f(n) = \sqrt{1+n\cdot\sqrt{1+(n+1)\cdot\sqrt{1+(n+2)\cdot\sqrt{1+(n+3)\cdot\sqrt{1+(n+4)\cdot\sqrt{\cdots}}}}}}$$

for easier reference. Notice that $f(n) = \sqrt{1+n\cdot f(n+1)}$.

Base case: If $n = 0$, then $f(n) = 1$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N} \cup \{0\}$, $f(n_0) = n_0 + 1$.

Induction step: Assuming $f(n_0) = n_0 + 1$, deduce that $f(n_0 + 1) = n_0 + 2$.

We know that $f(n_0) = n_0 + 1$, so squaring both sides gives

$$\begin{aligned} f(n_0)^2 &= (n_0+1)^2 \\ \Rightarrow 1 + n_0 \cdot f(n_0+1) &= n_0^2 + 2n_0 + 1 \\ \Rightarrow n_0 \cdot f(n_0+1) &= n_0(n_0+2) \\ \Rightarrow f(n_0+1) &= n_0 + 2. \end{aligned}$$

Thus, by the principle of mathematical induction, $f(n) = n + 1$ for all $n \in \mathbb{N} \cup \{0\}$. Setting $n = 1$ yields the original identity. □