Justify all answers!

(6 pts)

(1) [+3] What is wrong with the following proof that all integers are equal?

Claim. For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{Z}$ it must be the case that $a_1 = \cdots = a_n$.

Proof. Base case: For $n = 1$, and any $a_1 \in \mathbb{Z}$, $a_1 = a_1$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, and any $a_1, \ldots, a_{n_0} \in \mathbb{Z}$, it must be the case that $a_1 = \cdots = a_{n_0}$.

Induction step: Given the induction hypothesis, prove that for any $a_1, \ldots, a_{n_0+1} \in \mathbb{Z}$, we have $a_1 = \cdots = a_{n_0+1}$.

Let $a_1, \ldots, a_{n_0+1} \in \mathbb{Z}$ be arbitrary. First consider $a_1, \ldots, a_{n_0}$. As this is a set of n_0 integers, by the induction hypothesis, $a_1 = \cdots = a_{n_0}$. Also, $a_2, \ldots, a_{n_0+1}$ is a set of n_0 integers, so $a_2 = \cdots = a_{n_0+1}$. By transitivity, we have, $a_1 = a_2 = \cdots = a_{n_0} = a_{n_0+1}$.

Thus, as the statement is true for $n = 1$ and we have shown that if the statement is true for $n = n_0$, then it is true for $n = n_0 + 1$, by the principle of mathematical induction, all integers are equal. □

(2) [+3] What is wrong with the following proof that all natural numbers are equal?

Claim. For any $n \in \mathbb{N}$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n$, then $x = y$.

Proof. Base Case: For $n = 1$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = 1$, then it must be the case that $x = 1 = y$, so the base case holds.

Induction hypothesis: For some $n_0 \in \mathbb{N}$, if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n_0$, then $x = y$.

Induction step: Given the induction hypothesis, prove that if $x, y \in \mathbb{N}$ have $\max\{x, y\} = n_0 + 1$, then $x = y$.

Take any $x, y \in \mathbb{N}$ with $\max\{x, y\} = n_0 + 1$, then it is the case that $\max\{x - 1, y - 1\} = n_0$. By the induction hypothesis, we know that $x - 1 = y - 1$, so $x = y$.

Thus, as the statement is true for $n = 1$ and we have show that if the statement is true for $n = n_0$, then it is true for $n = n_0 + 1$, by the principle of mathematical induction, all natural numbers are equal. □