Justify all answers!

(12 pts)

(1) [+4] Let A, B, C, D be sets. Prove that, if $A \cup B \subseteq C \cup D$ and $C \subseteq A$ and $A \cap B = \varnothing$, then $B \subseteq D$.

Proof. We want to show that $B \subseteq D$, so let $x \in B$ be arbitrary; we must show that $x \in D$. Now, $x \in B$, so $x \in A \cup B$, so $x \in C \cup D$ as $A \cup B \subseteq C \cup D$. As we want to conclude that $x \in D$, it is enough to show that $x \notin C$. Well, $x \in B$ and $A \cap B = \varnothing$, so we must have $x \notin A$ or else $x \in A \cap B$. Thus, we must also have $x \notin C$ as otherwise $x \in A$ as $C \subseteq A$. Thus, $x \in C \cup D$ and $x \notin C$, so $x \in D$.

As $x \in B$ was arbitrary, we have shown $B \subseteq D$. □

(2) (a) [+3] Prove that if A, B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Let $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$ be arbitrary; we break into cases. First suppose $S \in \mathcal{P}(A)$, so $S \subseteq A$. As $A \subseteq A \cup B$, this means $S \subseteq A \cup B$, so $S \in \mathcal{P}(A \cup B)$. Similarly, if $S \in \mathcal{P}(B)$, then $S \subseteq B \subseteq A \cup B$, so $S \subseteq A \cup B$, so $S \in \mathcal{P}(A \cup B)$.

As $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$ was arbitrary, we have shown $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

(b) [+1] Must it be the case that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$?

Proof. No, not necessarily. Let $A = \{1\}$ and $B = \{2\}$, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}, \{2\}\}$ whereas $\mathcal{P}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. □

(3) [+4] Prove that $\bigcup_{n \in \mathbb{N}} [n] = \mathbb{N}$.

Proof. Let $S = \bigcup_{n \in \mathbb{N}} [n]$, so we may reference it more quickly. We must prove that $S \subseteq \mathbb{N}$ and $S \supseteq \mathbb{N}$.

($\subseteq$) Let $x \in S$ be arbitrary. By definition of the union, there is some $n \in \mathbb{N}$ for which $x \in [n]$. As $[n] = \{1, 2, \ldots, n\} \subseteq \mathbb{N}$, we have $x \in \mathbb{N}$. Therefore, as $x \in S$ was arbitrary, $S \subseteq \mathbb{N}$.

($\supseteq$) Let $x \in \mathbb{N}$ be arbitrary. Certainly $x \in [x]$, so $x \in S$ by definition of the union. As $x \in \mathbb{N}$ was arbitrary, $\mathbb{N} \subseteq S$. □

(Rewrite) [+3] Let A, B, C, D be sets. Prove or disprove: if $A \subseteq C$ and $B \subseteq D$, then $A \cap B \subseteq C \cap D$.

Proof. Let $x \in A \cap B$ be arbitrary. Thus $x \in A$ and $x \in B$. By assumption, $A \subseteq C$, so $x \in C$. Similarly, $B \subseteq D$, so $x \in D$. Thus, $x \in C \cap D$. As x was arbitrary, we have shown $A \cap B \subseteq C \cap D$. □