(9 pts)

(1) [+3] Letting $\mathbb{P}$ be the set of all prime numbers (note that $1 \notin \mathbb{P}$) and $2\mathbb{Z}$ be the set of all even integers, express the following statement using logical symbols: "There exists an even integer that can be written as the sum of two primes in two different ways."

Proof. $\exists x \in 2\mathbb{Z} \,.\, \exists p, q, r, s \in \mathbb{P} \,.\, (\{p, q\} \neq \{r, s\}) \wedge (x = p + q = r + s)$ ☐

(2) [+3] Prove or disprove the following: $\forall n \in \mathbb{Z} \,.\, (n < \sqrt{10}) \vee (n > 3)$.

Proof. As $p \vee q \equiv (\neg p) \Rightarrow q$, it suffices to prove the following: $\forall n \in \mathbb{Z} \,.\, (n \geq \sqrt{10}) \Rightarrow (n > 3)$. Now the statement is essentially trivial. Namely, if $n \in \mathbb{Z}$ has $n \geq \sqrt{10}$, then $n \geq \sqrt{10} > \sqrt{9} = 3$, so $n > 3$. ☐

(3) [+3] Consider the following two logical statements:

(a) $\forall \epsilon > 0 \,.\, \forall x, y \in \mathbb{R} \,.\, \exists \delta > 0 \,.\, (|x - y| < \delta) \Rightarrow (|(3x - 4) - (3y - 4)| < \epsilon)$

(b) $\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, \forall x, y \in \mathbb{R} \,.\, (|x - y| < \delta) \Rightarrow (|(3x - 4) - (3y - 4)| < \epsilon)$

Explain the difference between (a) and (b).

Proof. The only difference between these two statements in the order of the quantifiers. In (a), we have "$\forall x, y \in \mathbb{R} \,.\, \exists \delta > 0$," so the δ can depend on x and y. However, in (b), we have "$\exists \delta > 0 \,.\, \forall x, y \in \mathbb{R}$," so there must be some δ that works for every choice of x and y, i.e. δ is not allowed to depend on the choice of x and y.

In fact, (b)$\Rightarrow$(a), and it can be shown that both statements are true. As another note, if $f : \mathbb{R} \to \mathbb{R}$, then the statement "$\forall \epsilon > 0 \,.\, \forall x, y \in \mathbb{R} \,.\, \exists \delta > 0 \,.\, (|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \epsilon)$" is the definition of f being continuous, whereas "$\forall \epsilon > 0 \,.\, \exists \delta > 0 \,.\, \forall x, y \in \mathbb{R} \,.\, (|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \epsilon)$" is the definition of f being uniformly continuous. ☐

(Rewrite) [+2] Recall that $2\mathbb{Z}$ is the set of even integers. Write the following using logical symbols: "Every even integer can be written as $2m$ for some integer m."

Proof. $\forall n \in 2\mathbb{Z} \,.\, \exists m \in \mathbb{Z} \,.\, n = 2m$ ☐